PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 87, Number 2, February 1983

ON LATTICE SUMMING OPERATORS

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ABSTRACT. Given a Banach space E and a Banach lattice L, necessary and sufficient conditions on E and L are given such that every lattice summing operator $T: E \to L$ (cf. Introduction) is absolutely summing.

1. Introduction. The concept of absolutely summing operators has a certain natural analogue when the range space is a Banach lattice. Namely, an operator $T: E \to L$ is called lattice summing, if for every sequence (x_n) in E such that Σx_n converges unconditionally, the series $\Sigma |Tx_n|$ converges in L. Of course, if e.g. L is an L_1-space or E is an L_1-space and L is a Hilbert space, then both notions coincide.

The aim of this paper is to characterize all pairs E, L for which this happens.

In 1979 Yanovskii **[10]** investigated problems related to lattice summability. In particular, he formulated a conjecture that, if all lattice summing operators acting on the same Banach lattice L are absolutely summing, then L is isomorphic to L_1. However, it follows from our characterization that spaces of cotype 2 have this property. Another by-product of this paper is a still different formulation of the Dubinsky-Pełczyński-Rosenthal property $\Pi_2(\mathfrak{L}_\infty, E) = B(\mathfrak{L}_\infty, E)$.

2. Notations and preliminary facts. Throughout this paper E, F and L, K denote infinite-dimensional Banach spaces and Banach lattices, respectively. E' is the norm dual of E and p' stands for the exponent dual to $p \in [1, \infty]$. The natural basis in l_p, $1 \leq p < \infty$, is denoted by $e_1, e_2, \ldots$. For a finite sequence (x_i) in E we set

$$w_p(\{x_i\}) = \sup\left\{\left(\sum |\langle x', x_i\rangle|^p\right)^{1/p} : x' \in E', \|x'\| \leq 1\right\}.$$

An operator $T: E \to F$ is called p-absolutely summing if there exists a constant $C > 0$ such that, for all $x_1, \ldots, x_n \in E$,

$$\left(\sum \|Tx_i\|^p\right)^{1/p} \leq C w_p(\{x_i\}).$$

The smallest C is denoted by $\Pi_p(T)$ and is the norm in the Banach space $\Pi_p(E, F)$ of all p-absolutely summing operators.

An operator $T: E \to L$ is called lattice summing if, for some $C > 0$, $\|\Sigma |Tx_i|\| \leq C w_1(\{x_i\})$ for all $x_1, \ldots, x_n \in E$. The smallest C, denoted by $\Lambda(T)$, is the norm in the Banach space $\Lambda(E, L)$ of all lattice summing operators from E into L. Clearly $\Lambda(T) \leq \Pi_1(T)$.

Received by the editors October 19, 1981 and, in revised form, April 23, 1982 and June 22, 1982.

1980 *Mathematics Subject Classification*. Primary 47B10; Secondary 46B20.

Key words and phrases. p-absolutely summing operators, majorizing operators, Banach lattices, cotype of Banach spaces.

0002-9939/82/0000-0687/$03.25

The following easy construction shows how lattice summing operators lead to absolutely summing operators. Let (L, y'), where $y' \in L'$, $y \geq 0$, denotes the completion of L with respect to the seminorm $\langle y', |\circ|\rangle$. (L, y') is isomorphic to L_1 and the natural lattice homomorphism $I_{y'}\colon L \to (L, y')$ is continuous.

PROPOSITION 2.1. *The following assertions are equivalent for an operator $T\colon E \to L$:*

(i) *T is lattice summing;*

(ii) *for each $y' \geq 0$, $I_{y'} \circ T$ is absolutely summing;*

(iii) *for every, or equivalently, for just one infinite-dimensional L_1-space and for each positive operator $S\colon L \to L_1$, $S \circ T$ is absolutely summing.*

We omit a standard proof. In order to give examples of lattice summing operators we state the following result. First we recall that an operator $T\colon E \to L$ is said to be majorizing if

$$M(T) := \sup\left\{\left\| \sup_i |Tx_i| \right\| : \|x_i\| \leq 1, i = 1,\dots,n\right\}$$

is finite. We denote by $M(E, L)$ the Banach space of such operators.

PROPOSITION 2.2. *Let $T\colon E \to L$. T is lattice summing if and only if for each operator $U\colon l_\infty \to E$, $T \circ U$ is majorizing.*

PROOF. We have

$$\begin{aligned}
\Lambda(T) &= \sup_n \sup\left\{\left\| \sum_{i=1}^{n} |Tx_i| \right\| : w_1(\{x_i\}) \leq 1,\ \{x_i\} \subset E\right\} \\
&= \sup_n \sup\{\|\sup\{|Ta_i x_i| : |a_i| \leq 1\}\| : w_1(\{x_i\}) \leq 1\} \\
&= \sup_n \sup\{\|\sup\{|TSu| : u \in l_\infty^n,\ \|u\|_\infty \leq 1\}\| : S\colon l_\infty^n \to E,\ \|S\| \leq 1\} \\
&= \sup\{M(TS) : S\colon l_\infty \to E,\ \|S\| \leq 1\}. \qquad \text{Q.E.D.}
\end{aligned}$$

COROLLARY 2.1. *Each majorizing operator $T\colon E \to L$ is lattice summing and $\Lambda(T) \leq M(T)$. If $E = l_\infty$ we have more: $M(l_\infty, L) = \Lambda(l_\infty, L)$.*

Let us recall a few notions which play important roles in what follows.

By definition, E is of cotype q, where $q \geq 2$, if there exists a $C > 0$ such that for all finite sets $(x_i) \subseteq E$

$$\left(\sum \|x_i\|^q\right)^{1/q} \leq C\int_0^1 \left\|\sum r_i(t)x_i\right\| dt,$$

where $r_1, r_2, \dots$ are Rademacher functions. We say that L is p-concave, where $p \in [1, \infty)$, if for a $C > 0$,

$$\left(\sum \|x_i\|^p\right)^{1/p} \leq C\left\|\left(\sum |x_i|^p\right)^{1/p}\right\|$$

for all finite choices of $\{x_i\} \subseteq E$. Here

$$\left(\sum |x_i|^p\right)^{1/p} = \sup\left\{\sum a_i x_i : a_i \in \mathbf{R},\ \sum |a_i|^{p'} \leq 1\right\}$$

(cf. [3 or 5] for details). Note if L is 1-concave, then it is isomorphic to an L_1-space (cf. e.g. [9]). We say that F is finitely representable in E (respectively, K is lattice finitely representable in L) if there exists $\delta > 0$ such that, for each finite-dimensional subspace $F_0 \subseteq F$ (respectively, sublattice $K_0 \subseteq K$), one can find an isomorphism I_0: $F_0 \to F$ (respectively I_0: $K_0 \to L$) satisfying

$$\delta^{-1}\|x\| \leqslant \|I_0 x\| \leqslant \delta\|x\|$$

for all $x \in F_0$ (respectively, $x \in K_0$).

3. Main result.

THEOREM 3.1. *The following conditions are equivalent*:

(I) $\Lambda(E, L) = \Pi_1(E, L)$;

(II) $M(E, L) \subseteq \Pi_1(E, L)$;

(III) *there exists* $p \in [1,2]$ *such that both* $\Pi_{p'}(\mathfrak{L}_\infty, E) = B(\mathfrak{L}_\infty, E)$ *and* L *is* p*-concave*.

REMARK. We identify ∞-absolutely summing and bounded operators. The following is an adaptation of what we need from Rosenthal [8] and Maurey and Pisier [7].

PROPOSITION 3.1. *Let* $p \in (1,2]$. *The following properties of a Banach space* E *are equivalent*:

(i) $M(E, l_p) \subseteq \Pi_1(E, l_p)$;

(ii) *there exists a* $C > 0$ *such that*

$$\sum_j \Big(\sum_i |\langle x_i', x_j \rangle|^p \Big)^{1/p} \leqslant C w_1(\langle x_j \rangle) \big(\sum \|x_i'\|^p \big)^{1/p}$$

for all $x_1, \ldots, x_n \in E$ *and* $x_1', \ldots, x_m' \in E'$;

(iii) *for every Banach space* F, $\Pi_1(E, F) = \Pi_p(E, F)$;

(iv) $\Pi_{p'}(\mathfrak{L}_\infty, E) = B(\mathfrak{L}_\infty, E)$.

Moreover, if $p < 2$, *then each of the above conditions is equivalent to*

(v) $p' > q_E := \inf\{q\colon \Pi_q(\mathfrak{L}_\infty, E) = B(\mathfrak{L}_\infty, E)\}$.

The implications (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) are in Proposition 3 in [8]. The fact that infimum q_E is never attained for $q > 2$, which is precisely what (v) says, is due to Maurey and Pisier [7]. Finally, (ii) is just a reformulation of (i) expressed in terms of finite rank operators.

LEMMA 3.1. *Let* $M(E, L) \subseteq \Pi_1(E, L)$. *If* F *is finitely representable in* E *and* K *is lattice finitely representable in* L *then* $\Pi_1(T) \leqslant CM(T)$ *for all compact operators* T: $F \to K$, *where the constant* C *does not depend on the operators* T.

PROOF. Let M^c and Π_1^c denote the classes of compact operators in M and Π_1, respectively. A standard argument shows that $M(E, L) \subseteq \Pi_1(E, L)$ implies that $M^c(E, K) \subseteq \Pi_1^c(E, K)$. In particular, by the "local reflexivity principle" (cf. [4, Proposition I. 4]), $M^c(E, K'') \subseteq \Pi_1^c(E, K'')$. Finally, repeating the usual procedure, however, making use of the extension property of majorizing operators (as it is stated in [9, Proposition IV. 3.10]), we infer that $M^c(F, K) \subseteq \Pi_1^c(F, K)$. Q.E.D.

COROLLARY 3.1. *If l_q is finitely representable in E, where $1 < q \leqslant \infty$, and $M(E, L) \subseteq \Pi_1(E, L)$, then L is q'-concave. In particular, by Dvoretzky's theorem* [**2**], *$M(E, L) \subseteq \Pi_1(E, L)$ shows L to be 2-concave.*

PROOF. In fact, by Lemma 3.1 we may, and do, assume that $M(l_q, L) \subseteq \Pi_1(l_q, L)$; hence for some $C > 0$, $\Pi_1(T) \leqslant CM(T)$, for each operator of the form $T = \sum_{i=1}^n e_i \otimes y_i \colon l_q \to L$. But

$$\Pi_1(T) \geqslant \sup\left\{ \sum |a_i| \, \|y_i\| : w_1(\{a_i e_i\}) \leqslant 1 \right\} = \left(\sum \|y_i\|^{q'} \right)^{1/q'};$$

$$M(T) \leqslant \left\| \sup\left\{ \left| \sum \langle e_i, x \rangle y_i \right| : \|x\| \leqslant 1 \right\} \right\| = \left\| \left(\sum |y_i|^{q'} \right)^{1/q'} \right\|.$$

Combining these estimates we get the desired condition. Q.E.D.

PROOF OF THEOREM 3.1. (I) $\Rightarrow$ (II) is an immediate consequence of Proposition 2.2.

(II) $\Rightarrow$ (III). Let $p_L := \inf\{p \colon L \text{ is } p\text{-concave}\}$.
By [**4**, Theorem II. 2], l_{p_L} is lattice finitely representable in L. Hence, by Lemma 3.1 and Proposition 3.1, $\Pi_{p_L'}(\mathfrak{L}_\infty, E) = B(\mathfrak{L}_\infty, E)$.

Now we examine three cases: $p_L = 2$, $p_L \in (1,2)$ and $p_L = 1$. The first yields (III) with $p = 2$, in view of Corollary 3.1. In the second case we apply condition (v) of Proposition 3.1 so that $p_L' > q_E$ and this implies (III) with $p \in (1,2)$. $p_L = 1$ carries no information about E, however, if $q_E < \infty$, then again $q_E' > p_L = 1$ and we proceed as before in order to obtain (III) for a $p \in (1,2]$. On the other hand, if $q_E = \infty$ or, equivalently, if l_∞ is finitely representable in E (see [7]), then by Corollary 3.1, L is 1-concave (note that $p_L = 1$ does not imply 1-concavity, in general). This completes the proof of (II) $\Rightarrow$ (III).

(III) $\Rightarrow$ (I). Let $p \in [1,2]$ be such that $\Pi_{p'}(\mathfrak{L}_\infty, E) = B(\mathfrak{L}_\infty, E)$ and let L be p-concave. Hence, by Proposition 3.1, $\Pi_1(E, L) = \Pi_p(E, L)$. Therefore, by (III), there exist constants C and C_1 such that for each $T \colon E \to L$ we have

$$\Pi_1(T) \leqslant C\Pi_p(T) \leqslant CC_1 \sup\left\{ \left\| \left(\sum |Tx_i|^p \right)^{1/p} \right\| : w_p(\{x_i\}) \leqslant 1 \right\}$$

$$\leqslant CC_1 \sup\left\{ \left\| \int_0^1 \left| \sum Tx_i f_i(t) \right| dt \right\| : w_1\left(\sum x_i f_i \right) \leqslant 1 \right\}$$

where (f_i) is a sequence of independent standard real p-stable random variables on $[0, 1]$ and

$$w_1(f) = \sup\left\{ \int_0^1 |\langle x', f(t) \rangle| \, dt : x' \in E', \|x'\| \leqslant 1 \right\}$$

for E-valued function f (cf. [3]). By routine measure theory one can check that

$$\Lambda(T) = \sup \left\| \int_0^1 |Tf(t)| \, dt \right\|$$

where the supremum is taken over all finite rank measurable functions with $w_1(f) \leqslant 1$. Hence, $\Pi_1(T) \leqslant CC_1\Lambda(T)$ for all operators $T \colon E \to L$. This completes the proof of (III) $\Rightarrow$ (I). Q.E.D.

4. Comments and corollaries. By Maurey-Pisier [7], $q_E = \inf\{q\colon E \text{ is of cotype } q\}$. Hence, in virtue of Proposition 3.1, the third assertion of Theorem 3.1 can be rephrased in the following manner:

(III′) One of the following conditions is satisfied:

(i) l_∞ is finitely representable in E and L is isomorphic to L_1.

(ii) there exist p, q, $1 \leq p < q' \leq 2$, such that E is of cotype q and L is p-concave,

(iii) $\Pi_2(\mathcal{L}_\infty, E) = B(\mathcal{L}_\infty, E)$ and L is 2-concave.

If E is a Banach lattice, then the above takes a much simpler appearance (cf. [**6**]):

(III) There exist p, q such that either $p = q' = 1$ or $p = q' = 2$ or $1 \leq p < q' \leq 2$, and E is of cotype q and L is p-concave.

COROLLARY 4.1. $\Lambda(L, L) = \Pi_1(L, L)$ *if and only if* L *is* 2-*concave.*

COROLLARY 4.2. *For a Banach space* E, $\Pi_2(\mathcal{L}_\infty, E) = B(\mathcal{L}_\infty, E)$ *if and only if for some constant* $C > 0$ *for each operator* $U\colon E' \to l_1$,

$$\int_0^1 \Big\| \sum U x_i' r_i(t) \Big\| dt \leq C \|U\| \sup\Big\{ \Big(\sum \|V x_i'\|^2 \Big)^{1/2} : V\colon E' \to l_1, \|V\| \leq 1 \Big\}$$

for all $x_1', \ldots, x_n' \in E'$.

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