$L^p$-BOUNDEDNESS OF A CERTAIN CLASS OF MULTIPLIERS ASSOCIATED WITH CURVES ON THE PLANE. I

ALBERTO RUIZ

ABSTRACT. $L^p$-unboundedness for $p \neq 2$ is proved in the case of multipliers which are constant along curves. In particular $y = x^n$ is included in the range of curves.

While the sharpest result on Bochner-Riesz multipliers remains open in $\mathbb{R}^n$, $n \geq 3$, it was obtained in the plane by L. Carleson and P. Sjölin [1] and later by C. Fefferman [4] through different methods. A. Cordoba [2] gets this result as a consequence of some estimates for multipliers supported on thin sets ("bump functions") and his result allows one to guess the "blow up" as we approach boundary results: the problem is to take Kakeya-Nikodym's set, which becomes "the witness" of estimates in $\mathbb{R}^2$.

This "geometric decomposition" method allows us to obtain other results. First of all it is possible to get Sjölin's theorems [9] on Bochner-Riesz multipliers associated to more general curves than the circumference (see [8]).

W. Littman, C. McCarthy and N. M. Riviere [6] obtained nonboundedness results for the "Schrödinger equation" multiplier $1/(\xi_0 - (\xi_1^2 + \cdots + \xi_n^2) + i)$, namely for $4 < p < \infty$. Nevertheless, Riviere [7] conjectured it should be bounded in some $L^p$ with $p > 2$. The geometric method allows us to study a more general case; in the present paper we prove that the multiplier $\gamma(\xi_1 - \gamma(\xi_2))$ is only bounded in $L^2$, where $x = \gamma(y)$ is in a suitable class of curves in $\mathbb{R}^2$. The higher dimensional case follows from the present one after a theorem due to de Leeuw. The case $\gamma(x) = y^2$, which is just Riviere's, has been proved by C. Kenig and P. Tomas [5].

I would like to thank my teacher and friend Professor A. Cordoba for the orientation and advice that, in such a patient way, he has given to me concerning these problems.

We are going to refer to the class of curves $y = \gamma(x)$ such that:

(a) $y = \gamma(x)$ is $C^\infty(\mathbb{R})$ and $\gamma'(x) \to \infty$ as $x \to \infty$, and $\gamma''(x) > 0$ for large enough $x$.

(b) There exists a positive integer $k_0$ and constants $L_\gamma$ and $M_\gamma$ such that

$$1 < \frac{\gamma'(x_1)}{\gamma'(x_2)} < L_\gamma,$$

$$1 < \frac{\kappa(x_2)}{\kappa(x_1)} < M_\gamma$$

hold when $x_1$ and $x_2$ lie in $[2^k, 2^{k+1}]$ and $x_1 > x_2$ for every $k \geq k_0$; $\kappa(x)$ denotes the curvature of $\gamma$ at $x$.
(c) The length of the curve in the diadic interval \([2^k, 2^{k+1}]\) is larger than \(\kappa(2^k)^{-1/2}\) when \(k > k_0\).

The following curves satisfy the above hypotheses:

1. \(y = x^2, \alpha > 1,\)
2. \(y = x(\log x)^\alpha, \alpha > 0.\)

That is not the case of \(y(x) = e^x\), which does not satisfy (b). Consider \(\phi \in L^p \cap L^\infty(\mathbb{R})\) for some \(p > 1\), \(\phi\) a positive function increasing on \(\mathbb{R}^+\) and decreasing on \(\mathbb{R}^-\); take

\[
m(\xi_1, \xi_2) = \phi(\xi_2 - \gamma(\xi_1)), \quad \xi = (\xi_1, \xi_2) \in \mathbb{R}^2,
\]

and define

\[
\widehat{Tf}(\xi) = m(\xi)\hat{f}(\xi) \quad \text{for every } f \in S(\mathbb{R}^2).
\]

**Theorem 1.** \(T\) is a bounded operator \(L^p \to L^p \) iff \(p = 2\).

In order to prove Theorem 1 we need the following lemmas.

**Lemma 1 (Kakeya’s set).** Let \(\alpha, \mu, \tau\) be positive numbers. Then there exists a \(\delta_0 > 0\) such that given \(\delta < \delta_0\) there is a set \(E \subset \mathbb{R}^2\) and a family of pairwise disjoint rectangles \(\{R_j\}\) whose directions are in the interval \([-\alpha/2, \alpha/2]\) and such that

(i) their dimensions are \(\delta^{-1}/\mu \times \delta^{-1/2}\tau;\)

(ii) \(|E \cap \tilde{R}_j| \geq \min(1/20, \mu \tau/20\alpha) |R_j|\), where \(\tilde{R}_j\) is the usual adjacent rectangle to \(R_j;\)

(iii) \(|E| \leq (8\mu \tau/\alpha) |\log \delta|^{-1} |\log |\log \delta| |\cdot | \cup R_j|\).

Let us consider \(\lambda > 0\) and \(\Psi_\lambda^\lambda(\xi_1, \xi_2) = \chi_{[a_\lambda - \lambda, a_\lambda + \lambda]}(\xi_1, \xi_2)\)

where \(\{a_j\}\) is a sequence of real numbers. Given \(\{f_j\} \in L^p(l^2)\) we can state

**Lemma 2.** Assume \(\|Tf\|_p \leq C_p\|f\|_p\) for every \(f \in L^p\), then \(\|(\Sigma |T_j^\lambda f_j|^2)^{1/2}\|_p \leq C \cdot C_p\|(\Sigma |f_j|^2)^{1/2}\|_p\) with \(c\) an absolute constant, where

\[
(T_j^\lambda f)(\xi) = m(\xi)\Psi_j^\lambda(\xi)\hat{f}(\xi), \quad \xi = (\xi_1, \xi_2) \in \mathbb{R}^2.
\]
Its proof is immediate.

**Proof of Theorem I.** We can suppose, without loss of generality, that \( \phi \in L^1 \cap L^\infty(\mathbb{R}) \). If \( \|Tf\|_p \leq A\|f\|_p \), then \( \|\mathcal{S}f\|_p \leq A\|f\|_p \) with the same constant where
\[
\mathcal{S}f(\xi) = m(\xi)\chi_{[a,b] \times [\gamma(a),\gamma(b)]}(\xi_1,\xi_2)f(\xi).
\]

These \( a \) and \( b \) are chosen with the following properties:

(i) \( a = 2^\rho, a > b; b < 2^{\rho+1} \), where \( \rho \) is a positive integer to be fixed later on.

(ii) The length \( L \) of the arc of the curve from \( a \) to \( b \) is \( \kappa_1^{-1/2} \), where \( \kappa_1 \) is the curvature of \( \gamma \) at \( a \).

(iii) Let us consider a \( \delta > 0 \), to be fixed later on, only depending on \( \phi \) and \( \gamma \), and take \( \rho \) such that \( \delta \cdot \cos \alpha_1 < 1 \), where \( \alpha_1 \) is the angle between the tangent at \( a \) and the \( ox \)-axis.

(iv) \( \alpha_1 > \pi/4 \).

(v) \( 1 < \kappa_1/\kappa_2 < M_2 \) where \( \kappa_2 \) is the curvature at \( b \).

Consider the operator in Lemma 2 given by the strips whose direction is the normal to \( \gamma \) at the middle point of \([a,b]\), and \( \lambda = \delta^{1/2}\kappa_1^{-1/2} \) (the \( a_j \)'s will be fixed later on). There denote strips by \( B_j \),
\[
\left\| \left( \sum \left| T_j^\lambda f_j \right|^2 \right)^{1/2} \right\|_p \leq A \left\| \left( \sum |f_j|^2 \right)^{1/2} \right\|_p.
\]

We want to find some functions \( \{f_j\} \) which make \( \mathcal{A} \) nonbounded. Let us choose the parameters in Lemma 1, \( \delta = \cos \alpha_1, \mu = 8\xi, \tau = \kappa_1^{1/2}/8 \) and \( \alpha = L\kappa_2/2 \).
Let $R_j$ and $E$ be the suitably rotated rectangles and set in Lemma 1. Now we choose \( \{a_j\} \) such that the normal to the curve at \( x_j \) is the direction of \( R_j \), where \( x_j \) is the point on \( \gamma \) whose projection is \( a_j \). Since we chose \( \alpha = L\kappa_2/2 \) we are sure every \( x_j \) lies in the taken arc of the curve.

Let us remember the properties of \( \{R_j\} \) and \( E \).
(i) the dimensions of \( R_j \) are \( \delta^{-1}/8\kappa \times \delta^{-1}/2\kappa^{1/2}/8 \);
(ii) \( |E \cap R_j| \geq |R_j|/20 \);
(iii) \( |E| \leq 16\xi M_2(\log |\log \delta|/|\log \delta|) \cup R_j \).

Let us take \( f(x) = e^{ix} \chi_{R_j} \). We claim that \( |T^\lambda_{R_j f}(x)| \geq c(\gamma, \phi) \) for every \( x \in \bar{R}_j \),

\[
|T^\lambda_{R_j f}(x)| \geq \left| \int_{x \in R_j} e^{-i\phi} \chi_{R_j}(x) e^{ix(x-x')} \chi_{R_j}(x-y) \, dx \right|
\]

where \( R = [a, b] \times [\gamma(a), \gamma(b)] \).

For \( \xi \in R \) we have \( \xi = (\xi_1, \xi_2) = \Gamma(t) + s\eta(\Gamma(t)) \) where \( \Gamma(t) = (\Gamma_1(t), \Gamma_2(t)) \) is \( \gamma(t) \) parametrized by arclength and \( s \) is the normal distance.

Then

\[
(1.1) \quad |T^\lambda_{R_j f}(x)| \geq \left| \int_{\xi \in R} \phi(\xi_2 - \gamma(\xi_1)) \chi_{R_j}(\xi) \int_{\gamma \in x-R_j} \cos\langle x_j - \xi, y \rangle \, dy \, d\xi \right|
\]

\[
- \int_{\xi \in R} \phi(\xi_2 - \gamma(\xi_1)) \chi_{R_j}(\xi) \, dR_j \, d\xi.
\]

Let us look at the properties of \( \phi \) and \( \gamma \). In the case \( s > 0 \),

\[
\xi_2 - \gamma(\xi_1) \leq \frac{s}{\cos \alpha} \leq \frac{s}{\cos \alpha_2},
\]

\[
\xi_2 - \gamma(\xi_1) \geq \frac{s}{2 \cos \alpha_1}
\]

and for \( s < 0 \),

\[
\gamma(\xi_1) - \xi_2 \leq \frac{-2s}{\cos \alpha_2}, \quad \gamma(\xi_1) - \xi_2 \geq \frac{-s}{2 \cos \alpha_1}
\]

where \( \alpha_2 \) is the angle between the tangent at \( b \) and the ox-axis.
Therefore, when $s > 0$ we have

\[
\phi(\xi_2 - \gamma(\xi_1)) \geq \phi \left( \frac{s}{\cos \alpha_2} \right),
\]

\[
\phi(\xi_2 - \gamma(\xi_1)) \leq \phi \left( \frac{s}{2 \cos \alpha_1} \right),
\]

and similar relationships when $s < 0$.

Let us estimate $\cos(\langle x_j - \xi, y \rangle)$ when $y \in x - R_j$, $x \in \tilde{R}_j$ and $s \in [-\delta, \delta]$. $\chi_{[-\delta, \delta]}(s) \phi(\xi_2 - \gamma(\xi_1)) \chi_B(\xi)$ is supported in a rectangle whose dimensions are $4\xi \times 4\delta^{1/2}/\kappa^{1/2}$ since the curve is closer than $\delta$ to the tangent and the error due to the slope of the strip $B_j$ is at most $4\xi \delta < \delta^{1/2}/\kappa^{1/2}$ when $\delta$ is sufficiently small.

If

\[
\langle x_j - \xi, y \rangle = |x_j - \xi||y| \cos \left( \frac{\pi}{2} - \beta_1 - \beta_2 \right)
\]

\[
\leq |x_j - \xi||y| \sin \beta_1 \cos \beta_2 + \cos \beta_1 \sin \beta_2|
\]

\[
\leq 2\delta |y| \cos \beta_1 + \frac{4\delta^{1/2}}{\kappa^{1/2}} \cdot \frac{\sigma^{-1/2}\kappa^{1/2}}{8} < 1,
\]

then $\cos(\langle x_j - \xi, y \rangle) > 1/4$ and therefore, we can drop down the absolute value in (1.1) and consider two integrals, one in the case $s > 0$ and the other when $s < 0$. Both integrals can be bounded in a similar way so we only consider the case $s > 0$. (1.1) is bounded below by

\[
\left\{ \frac{1}{4} \int_{s \in [0, \delta]} \phi \left( \frac{s}{\cos \alpha_2} \right) \chi_B(\xi) \, d\xi - \int_{s > \delta} \phi \left( \frac{s}{2 \cos \alpha_1} \right) \chi_B(\xi) \, d\xi \right\} |R_j|
\]

\[
\geq |R_j| \left\{ \frac{1}{4} \int_{s \in [0, \delta]} \phi \left( \frac{s}{\cos \alpha_2} \right) \chi_B(\Psi(s, t))(1 + s\kappa(t)) \, ds \, dt
\]

\[
- \int_{s > \delta} \phi \left( \frac{s}{2 \cos \alpha_1} \right) \chi_B(\Psi(s, t))(1 + s\kappa(t)) \, ds \, dt \right\};
\]
after changing
\[ \xi = \Psi(s, t) = \frac{1}{64\pi} \left\{ \frac{1}{8L_\theta} \int_{u \in [0, \xi L_\theta]} \phi(u) \, du - 16 \int_{u \geq \xi/2} \phi(u) \, du \right\} \]
by taking \( \nu = L_\gamma 16^2 / (L_\gamma 16^2 + 1) \), \( 0 < \nu < 1 \), and \( \xi \) such that
\[ \min \left\{ \int_{u \in [0, \xi L_\theta]} \phi(u) \, du, \int_{u \in [0, \xi/2]} \phi(u) \, du \right\} \geq \nu \int_0^\infty \phi(u) \, du. \]
Then
\[ |T_\gamma f_j(x)| \geq \frac{1}{64\pi} \frac{\|\phi\|_1}{26L_\gamma} = c(\gamma, \phi) \quad \text{if} \ x \in \tilde{R}_j. \]

We apply the methods in [4] and obtain
\[ A > cM_{\gamma}^{-\frac{(p-2)}{2}} \frac{\log |\sigma|}{\log \log |\delta|}^{\frac{(p-2)}{2p}}. \]
Since \( \delta = \cos \alpha_1 \), it suffices to choose \( \rho \to -\infty \) to get \( A \to \infty \).

REFERENCES


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MINNESOTA 55455