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NOTE ON RESTRICTION OF FOURIER TRANSFORMS

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ABSTRACT. A technique for obtaining necessary conditions on restriction of Fourier transforms is introduced.

In [1], E. Prestini has proved that if α is a compact C^3 curve in $\mathbf{R}^3$ with nonvanishing curvature and torsion, then the inequality

$$(1) \qquad \|\hat{f}|_\alpha\|_{L^q(\alpha)} \leqslant C_p \|f\|_{L^p(\mathbf{R})^3}, \qquad f \in \mathcal{S},$$

holds if $1 \leqslant p < 15/13$ and $1/q > 6(1 - 1/p)$. The inequality does not hold if $p \geqslant 6/5$ or $1/q < 6(1 - 1/p)$. In this note we shall elaborate an idea of Knapp to prove that the inequality does not hold if $p > 7/6$. Our argument, which can be applied in similar situations also, is presented in the following paragraph.

We assume α is defined by the equation $(t, \phi(t), \psi(t))$, $0 \leqslant t \leqslant \eta$, where η is a small positive number and $\phi(t) = t^2 + \xi(t)$, $\psi(t) = t^3 + \zeta(t)$, $\xi(t)$ and $\zeta(t)$ are infinitesimals of third and fourth order w.r.t. t. Choose a large positive number M. For each positive integer k, set $\eta_k = 2^{-k}\eta$ and $\delta_k = \eta_k/M$. For each $j = 1, 2, \ldots, 2^k - 1$, let $Q_{k,j}$ be the parallellepiped centered at $\alpha(j\eta_k)$ and whose dimensions are δ_k, δ_k^2, δ_k^3 along the tangent, normal and binormal at $\alpha(j\eta_k)$, respectively. Note that for M sufficiently large, we may asume that, for each k, the collection $\{(1 + 1/M)Q_{k,j}: 0 < j < 2^k\}$ are pairwise disjoint and there exists $\theta > 0$ such that, for each k and j, $\{\alpha(t): |t - j\eta_k| < \theta\delta_k\} \subset Q_{k,j}$. Choose a smooth function g such that $g(x) = 1$ if x lies in Q, the unit cube centered at the origin, and $g(x) = 0$ if x lies outside $(1 + 1/M)Q$. Put $g_k(x_1, x_2, x_3) = g(x_1/\delta_k, x_2/\delta_k^2, x_3/\delta_k^3)$. Performing a suitable rigid motion to g_k, we obtain a function $g_{k,j}$ such that $g_{k,j}(x) = 1$ if x lies in $Q_{k,j}$ and $g_{k,j}(x) = 0$ if x lies outside $(1 + 1/M)Q_{k,j}$. Let $\hat{f}_{k,j} = g_{k,j}$. Then $\|f_{k,j}\|_{L^p(\mathbf{R}^3)} = \delta_k^{6(1-1/p)}\|\hat{g}\|_{L^p(\mathbf{R}^3)}$. Since the functions $f_{k,j}$, $0 < j < 2^k$, are rapidly decreasing, there exists points $w_{k,j}$, $0 < j < 2^k$, such that

$$\left\|\sum_j \tau_{w_{k,j}} f_{k,j}\right\|^p_{L^p(\mathbf{R}^3)} \leqslant 2\sum_j \left\|\tau_{w_{k,j}} f_{k,j}\right\|^p_{L^p(\mathbf{R}^3)}.$$

Note that, for $|t - j\eta_k| < \theta\delta_k$ and $0 < j < 2^k$,

$$\left|\widehat{\sum_j \tau_{w_{k,j}} f_{k,j}}(\alpha(t))\right| = 1.$$

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If the inequality (1) is true, we would have

$$\sum_j \int_{j\eta_k-\theta\delta_k}^{j\eta_k+\theta\delta_k} 1 \leq C\Big(\sum_j \delta_k^{6(p-1)}\Big)^{q/p}.$$

Hence $1 \leq C \cdot 2^k \cdot 2^{-6(p-1)k}$, for all positive integers k. Clearly this can hold only if $p \leq 7/6$.

REMARK. The above idea is useful even if the curvature vanishes somewhere (see **[2]**).

REFERENCES

1. E. Prestini, *A restriction theorem for space curves*, Proc. Amer. Math. Soc. **70** (1978), 8–10.
2. M. C. Hu, *Restriction of Fourier transforms to plane curves* (preprint).

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