NOTE ON RESTRICTION OF FOURIER TRANSFORMS

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Abstract. A technique for obtaining necessary conditions on restriction of Fourier transforms is introduced.

In [1], E. Prestini has proved that if \( \alpha \) is a compact \( C^3 \) curve in \( \mathbb{R}^3 \) with nonvanishing curvature and torsion, then the inequality

\[
\| \hat{f}_\alpha \|_{L^q(\alpha)} \leq C \| f \|_{L^p(\mathbb{R}^3)}, \quad f \in \mathcal{S},
\]

holds if \( 1 < p < 15/13 \) and \( 1/q > 6(1 - 1/p) \). The inequality does not hold if \( p \geq 6/5 \) or \( 1/q < 6(1 - 1/p) \). In this note we shall elaborate an idea of Knapp to prove that the inequality does not hold if \( p > 7/6 \). Our argument, which can be applied in similar situations also, is presented in the following paragraph.

We assume \( \alpha \) is defined by the equation \((t, \phi(t), \psi(t))\), \( 0 \leq t \leq \eta \), where \( \eta \) is a small positive number and \( \phi(t) = t^2 + \xi(t) \), \( \psi(t) = t^3 + \zeta(t) \), \( \xi(t) \) and \( \zeta(t) \) are infinitesimals of third and fourth order w.r.t. \( t \). Choose a large positive number \( M \).

For each positive integer \( k \), set \( \eta_k = 2^{-k} \eta \) and \( \delta_k = \eta_k/M \). For each \( j = 1, 2, \ldots, 2^k - 1 \), let \( Q_{k,j} \) be the parallelepiped centered at \( \alpha(j\eta_k) \) and whose dimensions are \( \delta_k \), \( \delta_k^2 \), \( \delta_k^3 \) along the tangent, normal and binormal at \( \alpha(j\eta_k) \), respectively. Note that for \( M \) sufficiently large, we may assume that, for each \( k \), the collection \( \{(1 + \sqrt{M})Q_{k,j} : 0 < j < 2^k\} \) are pairwise disjoint and there exists \( \theta > 0 \) such that, for each \( k \) and \( j \), \( \{\alpha(t) : |t - j\eta_k| < \theta \delta_k\} \subset Q_{k,j} \). Choose a smooth function \( g \) such that \( g(x) = 1 \) if \( x \) lies in \( Q \), the unit cube centered at the origin, and \( g(x) = 0 \) if \( x \) lies outside \( (1 + \sqrt{M})Q \). Put \( g_k(x_1, x_2, x_3) = g(x_1/\delta_k, x_2/\delta_k^2, x_3/\delta_k^3) \). Performing a suitable rigid motion to \( g_k \), we obtain a function \( g_{k,j} \) such that \( g_{k,j}(x) = 1 \) if \( x \) lies in \( Q_{k,j} \) and \( g_{k,j}(x) = 0 \) if \( x \) lies outside \( (1 + 1/M)Q_{k,j} \). Let \( \hat{g}_{k,j} = g_{k,j} \). Then

\[
\| f_{k,j} \|_{L^p(\mathbb{R}^3)} = \delta_k^{k(1-1/p)} \| \hat{g}_{k,j} \|_{L^p(\mathbb{R}^3)}.
\]

Since the functions \( f_{k,j} \), \( 0 < j < 2^k \), are rapidly decreasing, there exists points \( w_{k,j} \), \( 0 < j < 2^k \), such that

\[
\left\| \sum_j \tau_{w_{k,j}} f_{k,j} \right\|_{L^p(\mathbb{R}^3)}^p \leq 2 \sum_j \left\| \tau_{w_{k,j}} f_{k,j} \right\|_{L^p(\mathbb{R}^3)}^p.
\]

Note that, for \( |t - j\eta_k| < \theta \delta_k \) and \( 0 < j < 2^k \),

\[
\left| \sum_j \tau_{w_{k,j}} f_{k,j}(\alpha(t)) \right| = 1.
\]
If the inequality (1) is true, we would have

\[ \sum \int_{\eta_k+\theta\delta_k}^{\eta_k-\theta\delta_k} 1 \leq C \left( \sum \delta_k \right)^{a/p}. \]

Hence \( 1 \leq C \cdot 2^k \cdot 2^{-6(p-1)k} \), for all positive integers \( k \). Clearly this can hold only if \( p \leq 7/6 \).

REMARK. The above idea is useful even if the curvature vanishes somewhere (see [2]).

REFERENCES

2. M. C. Hu, Restriction of Fourier transforms to plane curves (preprint).

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