**H*(MO(8); Z/2) IS AN EXTENDED A₂*-COALGEBRA**

**DAVID J. PENGELLEY**

**Abstract.** We show that $H^*(MO(8); Z/2)$ is an extended $A^*_2$-coalgebra, where $A^*_2$ is the subalgebra of the Steenrod algebra generated by \{Sq^1, Sq^2, Sq^4\}. The method yields an analogous result for $H^*(M \text{Spin}; Z/2)$.

Recently Don Davis conjectured [D] that $H^*(MO(8); Z/2)$ is an extended $A^*_2$-coalgebra and discussed various consequences of such a result. We apply the method introduced in [P] to prove his conjecture.

Let $A^*_2 \subset A^*$ be the subalgebra generated by \{Sq^1, Sq^2, Sq^4\}.

**Theorem A.** $H^*MO(8)$ is an extended $A^*_2$-coalgebra, i.e., there is an $A^*$-coalgebra $N$ such that $H^*MO(8) \cong A^* \otimes A^*_2 N$ as an $A^*$-coalgebra.

**Corollary (Bahri and Mahowald [BM]).** $A^*/A^*_2$ is a direct summand in $H^*MO(8)$.

**Theorem B.** $H^*M \text{Spin}$ is an extended $A^*_1$-coalgebra.

**Proof of Theorem A.** Let $p : BO(8) \to BO$ be the covering map. Recall [S] that $p^* : H^*BO \to H^*BO(8)$ is onto, and $H^*BO(8) = Z/2[p^*w_n; \alpha(n - 1) \geq 3]$, where \(\alpha(m)\) is the number of ones in the dyadic expansion of $m$. $H^*BO(8)$ is a sub-Hopf algebra of $H^*_BO$, and hence by Borel's theorem is also polynomial [B].

Let $p_j$ be the coalgebra primitive in $H_{2\cdot -1} BO$ (and, via the Thom isomorphism, in $H_{2\cdot -1} MO$). From the inductive formula for Newton polynomials, $p_j$ is indecomposable in $H^*_MO$. So we can consider the polynomial subalgebra

$$P_2 = Z/2[p^8, p^4, p^2, p_4, \ldots] \subset H^*_MO.$$  

The map $QH^*BO \to QH^*BO(8)$ of indecomposable quotients is an isomorphism if $\alpha(n - 1) \geq 3$; and thus so is the map $PH^*_BO(8) \to PH^*_BO$ of coalgebra primitives. So the generators of $P_2$ lie in $H^*_MO(8)$, and thus all of $P_2$ does.

The coaction $\psi$ on $P_2 \subset H^*_MO(8)$ is known [BP]: $\psi p_j = \Sigma_i \xi_i \otimes p^i$. Since $A_2 = (A^*_2)^* = A/(\xi_1, \xi_2, \xi_3, \xi_4, \ldots)$, the augmentation ideal $P_2$ is clearly a submodule of $H^*_MO(8)$ over $A_2$ (although not over $A$). Thus so is the ideal $I$ generated by $P_2$ in $H^*_MO(8)$.

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Therefore by coassociativity we can form the diagram

\[
\begin{align*}
H_\ast MO(8) & \xrightarrow{\psi} A \otimes H_\ast MO(8) \xrightarrow{1 \otimes \pi} A \otimes H_\ast MO(8) / I \\
\vdash & \quad \\
\quad & \\
& \quad A \square_{A_3} H_\ast MO(8) / I
\end{align*}
\]

of $A$-algebras (the right three are here $A$-comodules using only the coproduct in the left factor $A$).

Since $H_\ast MO(8)$ has rank one if $m = 8, 12, 14$, and $p_n$ is indecomposable in $H_\ast MO$, \{ $p_8^8, p_8^2, p_7^2, p_4, \ldots$ \} are polynomial generators for $H_\ast MO(8)$ in their degrees. Now $(1 \otimes \pi) \circ \psi$ maps $p_1^8$ to $\xi_1^8 \otimes 1$, $p_2^4$ to $\xi_2^4 \otimes 1$, $p_3^2$ to $\xi_3^2 \otimes 1$, and $p_j$ to $\xi_j \otimes 1$ for $j \geq 4$, so $(1 \otimes \pi) \circ \psi$ is monic.

Finally, since $A$ is a sum of $A_3$'s as a right $A_2$ comodule,

\[
A \square_{A_3} H_\ast MO(8) / I \cong (A \square_{A_2} Z/2) \otimes H_\ast MO(8) / I
\]
as a graded vector space, and the latter in turn has the same graded rank as $H_\ast MO(8)$ since $A \square_{A_2} Z/2 = Z/2[\xi_1^8, \xi_2^4, \xi_3^2, \xi_4^2, \ldots]$. So $\psi$ is an $A$-algebra isomorphism. $\square$

Proof of Corollary. $H_\ast MO(8) / I$ begins in dimension 16, but $A_3^8$ is generated by \{ $Sq^1, Sq^2, Sq^4$ \}, so $Z/2$ in dimension zero is a split $A_2$ summand of $H_\ast MO(8) / I$. $\square$

The theorem for $M$ Spin has an identical proof, with $\alpha(n - 1) \geq 3$ replaced by $\alpha(n - 1) \geq 2$, and $A_2$ by $A_1$.

Don Davis has pointed out that since $P_2$ in $H_\ast BO$ is the image of $H_\ast \Omega^2 \Sigma^2 BO(8)$ under the Bahri-Mahowald map [BM], it follows that the $A_2^8$-coalgebra structure on $N$ is the restriction of an unstable $A^*$-coalgebra action, and thus $H^*(MO(8); Z/2)$ is isomorphic to $A^*/A_2^8 \otimes N$ with diagonal $A^*$ action.

References


[S] R. E. Stong, Determination of $H^\ast (BO(k, \ldots, \infty); Z_2)$ and $H^\ast (BU(k, \ldots, \infty); Z_2)$, Trans. Amer. Math. Soc. 107 (1963), 526–544.

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

Current address: Department of Mathematical Sciences, New Mexico State University, Las Cruces, New Mexico 88003