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$H^*(MO\langle 8\rangle; Z/2)$ IS AN EXTENDED A_2^*-COALGEBRA

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ABSTRACT. We show that $H^*(MO\langle 8\rangle; Z/2)$ is an extended A_2^*-coalgebra, where A_2^* is the subalgebra of the Steenrod algebra generated by $\{\mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^4\}$. The method yields an analogous result for $H^*(M\mathrm{Spin}; Z/2)$.

Recently Don Davis conjectured **[D]** that $H^*(MO\langle 8\rangle; Z/2)$ is an extended A_2^*-coalgebra and discussed various consequences of such a result. We apply the method introduced in **[P]** to prove his conjecture.

Let $A_2^* \subset A^*$ be the subalgebra generated by $\{\mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^4\}$.

THEOREM A. *$H^*MO\langle 8\rangle$ is an extended A_2^*-coalgebra, i.e., there is an A_2^*-coalgebra N such that $H^*MO\langle 8\rangle \cong A^* \otimes_{A_2^*} N$ as an A^*-coalgebra.*

COROLLARY (BAHRI AND MAHOWALD **[BM]**). *$A^*//A_2^*$ is a direct summand in $H^*MO\langle 8\rangle$.*

THEOREM B. *$H^*M\mathrm{Spin}$ is an extended A_1^*-coalgebra.*

PROOF OF THEOREM A. Let $p\colon BO\langle 8\rangle \to BO$ be the covering map. Recall **[S]** that $p^*\colon H^*BO \to H^*BO\langle 8\rangle$ is onto, and $H^*BO\langle 8\rangle = Z/2[p^*w_n\colon \alpha(n-1) \geq 3]$, where $\alpha(m)$ is the number of ones in the dyadic expansion of m. $H_*BO\langle 8\rangle$ is a sub-Hopf algebra of H_*BO, and hence by Borel's theorem is also polynomial **[B]**.

Let p_j be the coalgebra primitive in $H_{2^j-1}BO$ (and, via the Thom isomorphism, in $H_{2^j-1}MO$). From the inductive formula for Newton polynomials, p_j is indecomposable in H_*MO. So we can consider the polynomial subalgebra

$$P_2 = Z/2[p_1^8, p_2^4, p_3^2, p_4, \ldots] \subset H_*MO.$$

The map $QH^nBO \xrightarrow{p^*} QH^nBO\langle 8\rangle$ of indecomposable quotients is an isomorphism if $\alpha(n-1) \geq 3$; and thus so is the map $PH_nBO\langle 8\rangle \xrightarrow{p_*} PH_nBO$ of coalgebra primitives. So the generators of P_2 lie in $H_*MO\langle 8\rangle$, and thus all of P_2 does.

The coaction ψ on $P_2 \subset H_*MO\langle 8\rangle$ is known **[BP]**: $\psi p_j = \sum_i \zeta_i \otimes p_{j-i}^{2^i}$. Since $A_2 = (A_2^*)^* = A/(\zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4, \ldots)$, the augmentation ideal $\overline{P_2}$ is clearly a subcomodule of $H_*MO\langle 8\rangle$ over A_2 (although not over A). Thus so is the ideal I generated by $\overline{P_2}$ in $H_*MO\langle 8\rangle$.

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Therefore by coassociativity we can form the diagram

$$\begin{tikzcd} H_*MO\langle 8\rangle \arrow[r, "\psi"] \arrow[drr, "\hat{\psi}"'] & A \otimes H_*MO\langle 8\rangle \arrow[r, "1\otimes\pi"] & A \otimes H_*MO\langle 8\rangle/I \\ & & A \,\square_{A_2} H_*MO\langle 8\rangle/I \arrow[u, phantom, "\cup"] \end{tikzcd}$$

of A-algebras (the right three are here A-comodules using only the coproduct in the left factor A).

Since $H_m MO\langle 8\rangle$ has rank one if $m = 8, 12, 14$, and p_n is indecomposable in H_*MO, $\{p_1^8, p_2^4, p_3^2, p_4, \ldots\}$ are polynomial generators for $H_*MO\langle 8\rangle$ in their degrees. Now $(1 \otimes \pi) \circ \psi$ maps p_1^8 to $\zeta_1^8 \otimes 1$, p_2^4 to $\zeta_2^4 \otimes 1$, p_3^2 to $\zeta_3^2 \otimes 1$, and p_j to $\zeta_j \otimes 1$ for $j \geq 4$, so $(1 \otimes \pi) \circ \psi$ is monic.

Finally, since A is a sum of A_2's as a right A_2 comodule,

$$A \,\square_{A_2} H_*MO\langle 8\rangle/I \cong \left(A \,\square_{A_2} Z/2\right) \otimes H_*MO\langle 8\rangle/I$$

as a graded vector space, and the latter in turn has the same graded rank as $H_*MO\langle 8\rangle$ since $A \,\square_{A_2} Z/2 = Z/2[\zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4, \ldots]$. So $\hat{\psi}$ is an A-algebra isomorphism. $\square$

Proof of Corollary. $\overline{H_*MO\langle 8\rangle/I}$ begins in dimension 16, but A_2^* is generated by $\{\mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^4\}$, so $Z/2$ in dimension zero is a split A_2 summand of $H_*MO\langle 8\rangle/I$. $\square$

The theorem for $M\mathrm{Spin}$ has an identical proof, with $\alpha(n-1) \geq 3$ replaced by $\alpha(n-1) \geq 2$, and A_2 by A_1.

Don Davis has pointed out that since P_2 in H_*BO is the image of $H_*\Omega^2\Sigma^2 BO\langle 8\rangle^{15}$ under the Bahri-Mahowald map **[BM]**, it follows that the A_2^*-coalgebra structure on N is the restriction of an unstable A^*-coalgebra action, and thus $H^*(MO\langle 8\rangle; Z/2)$ is isomorphic to $A^*//A_2^* \otimes N$ with diagonal A^* action.

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