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ON A CERTAIN CLASS OF M_1-SPACES

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ABSTRACT. Let $\mathfrak{P}$ be the class of all M_1-spaces whose every closed subset has a closure-preserving open neighborhood base. A characterization is given, and it is proved that the adjunction space $X \cup_f Y$ is an M_1-space if $X \in \mathfrak{P}$ and Y is an M_1-space. Moreover, it is proved that if X is a space such that for each metrizable space Y, every closed subspace of $X \times Y$ is an M_1-space, then $X \in \mathfrak{P}$.

1. Introduction. Let (P_i), $i = 1, 2, 3, 4$, be the following statements concerning M_1-spaces [**3**] and stratifiable spaces [**1**]:

(P_1) Every stratifiable space is an M_1-space.

(P_2) Each closed subspace of an M_1-space is also an M_1-space.

(P_3) Each adjunction space of M_1-spaces is also an M_1-space.

(P_4) Each closed subset of an M_1-space has a σ-closure-preserving open neighborhood base.

The problems whether the (P_i) are true or not are still open. The problems of (P_1), (P_2) and (P_3) are posed by Ceder [**3**]. On the other hand, the problem of (P_4) is posed by Borges and Lutzer in [**2**], where they characterized stratifiable spaces (which coincide with M_2-spaces by [**4** and **8**]) as paracompact σ-spaces such that each closed subset has a σ-closure-preserving neighborhood base which consists of not necessarily open neighborhoods. By Borges [**1**] and by the recent result of Heath and Junnila [**6**], it follows that $(\mathrm{P}_1) \rightleftarrows (\mathrm{P}_2) \to (\mathrm{P}_3) \to (\mathrm{P}_4)$. As is well known by Ceder [**3**, Lemma 7.3], every closed subset of an M_2-space has a closure-preserving neighborhood base. Corresponding to this, let $(\mathrm{P}_4)'$ be the statement as follows:

$(\mathrm{P}_4)'$ Every closed subset of an M_1-space has a closure-preserving open neighborhood base.

Obviously $(\mathrm{P}_4)' \to (\mathrm{P}_4)$. The author in this paper studies the class $\mathfrak{P}$ of M_1-spaces satisfying $(\mathrm{P}_4)'$ and proves that if (P_2) is assumed to be true, then $(\mathrm{P}_4)'$ holds.

In this paper, all spaces are assumed to be regular and Hausdorff and all mappings to be continuous and onto unless the contrary is stated explicitly. N always denotes the positive integers. To state the results and the proofs, we give the definitions to the terminology used below. A mapping $f\colon X \to Y$ is said to be *irreducible* if no closed proper subspace of X is mapped onto Y by f. A space X is said to be *monotonically normal* [**7**] if the following is satisfied: To each pair (H, K)

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of separated subsets of X one can assign an open set $D(H, K)$ in such a way that

(i) $H \subset D(H, K) \subset \overline{D(H, K)} \subset X - K$, and

(ii) if (H', K') is a pair of separated sets having $H \subset H'$ and $K' \subset K$, then $D(H, K) \subset D(H', K')$.

According to [7], every stratifiable space is monotonically normal, and this fact is used below.

2. The class $\mathcal{P}$.

DEFINITION 1. Let $\mathcal{P}$ be the class of all M_1-spaces satisfying $(\mathrm{P}_4)'$.

Most common examples of M_1-spaces seem to belong to $\mathcal{P}$. For example: (1) stratifiable F_σ-metrizable spaces due to Gruenhage [5], (2) M_1-spaces with $\dim \leqslant 0$ and (3) the closed image of an M_0-space (a space which has a σ-closure-preserving base consisting of closed and open sets) belong to $\mathcal{P}$. The proofs are given below implicitly. Every Lašnev space, and more generally every L-space of Nagami [9], belongs to $\mathcal{P}$, but the converse is not true as shown by [9, Examples 2.1, 2.2].

LEMMA 1. *Let M be a closed subspace of a stratifiable space X. Then to each open set U of M, we can assign an open set U' of X such that $U' \cap M = U$ and the following is satisfied*:

(*) *$U' \cap V' = \varnothing$ whenever U, V are disjoint open sets of M.*

PROOF. Let $\mathcal{F} = \bigcup \{\mathcal{F}_n : n \in N\}$ be a σ-locally finite closed network of M such that each $\mathcal{F}_n$ is locally finite and $\mathcal{F}_n \subset \mathcal{F}_{n+1}$. For each open set W of M and each $n \in N$, set

$$\mathcal{F}_n(W) = \{F \in \mathcal{F}_n : F \subset W\}, \quad \mathcal{F}_n(W)^\# = \bigcup \{F : F \in \mathcal{F}_n(W)\}.$$

We shall construct U' for an arbitrary open set U of M by

$$U_n = D\big(\mathcal{F}_n(U)^\#, M - U\big) - \overline{D\big(\mathcal{F}_n(M - \overline{U})^\#, \overline{U}\big)}, \qquad n \in N,$$
$$U' = \bigcup \{U_n : n \in N\}.$$

Obviously $U' \cap M = U$. To see (*) for disjoint open sets U, V of M, assume that $V_m \cap U_n \neq \varnothing$, $m, n \in N$. Without loss of generality, we can assume $n \geqslant m$. Observe

$$D\big(\mathcal{F}_m(V)^\#, M - V\big) \subset D\big(\mathcal{F}_n(M - \overline{U})^\#, \overline{U}\big).$$

This contradicts the fact $V_m \cap U_n \neq \varnothing$.

In the above proof, if we use only the fact that X is monotonically normal, we obtain the following:

COROLLARY. *Let M be a closed subspace of a monotonically normal space X and let $\{U_\alpha : \alpha \in A\}$ be an open family of M. Then there exists an open family $\{U'_\alpha : \alpha \in A\}$ of X such that $U'_\alpha \cap M = U_\alpha$ for each $\alpha \in A$ and for each $B \subset A$*

$$\overline{\bigcup \{U_\alpha : \alpha \in B\}} = \overline{\bigcup \{U'_\alpha : \alpha \in B\}} \cap M.$$

In fact, if we define

$$U'_\alpha = \bigcup \{D(\{x\}, M - U_\alpha) : x \in U_\alpha\}, \qquad \alpha \in A,$$

then $\{U'_\alpha : \alpha \in A\}$ satisfied the required property.

DEFINITION 2. (NAGATA [**10**, DEFINITION 2]). A space X is said to have the *property* (ECP) when the following is satisfied: If $X' = X \cup F$, where X, F are closed in X', and if $\mathfrak{U} = \{U_\alpha: \alpha \in A\}$ is a closure-preserving open collection in the subspace F, then for each $\alpha \in A$ there is a family $\{U'_\beta: \beta \in B_\alpha\}$ of open sets of X' satisfying (1) $\mathfrak{U}' = \{U'_\beta: \beta \in \bigcup\{B_\alpha: \alpha \in A\}\}$ is closure-preserving in X', (2) for each $\beta \in B_\alpha$, $U'_\beta \cap F = U_\alpha$ and for every open set V in X' with $V \cap F = U_\alpha$, there is $\beta \in B_\alpha$ such that $U_\alpha \subset U'_\beta \subset V$, (3) for every open set W of F, there is an open set W' of X' such that $W' \cap F = W$ and such that $W' \cap U'_\beta = \varnothing$ whenever $\beta \in B_\alpha$ and $W \cap U_\alpha = \varnothing$.

THEOREM 1. *The following are equivalent for a stratifiable space X:*

(1) $X \in \mathfrak{P}$.

(2) *For each closed subspace M of X, there exists an open family $\mathfrak{U}$ of X such that $\mathfrak{U}$ is closure-preserving in $X - M$ and for each open set V of X there exists $U \in \mathfrak{U}$ such that $U \subset V$ and $U \cap M = V \cap M$.*

(3) *X has* (ECP).

PROOF. (3) $\to$ (1) is trivial.

(1) $\to$ (2): Let $\{G_n: n \in N\}$ be a sequence of open sets of X such that $M = \bigcap\{G_n: n \in N\}$, $\overline{G}_{n+1} \subset G_n$, $n \in N$. Let $\mathfrak{F} = \bigcup\{\mathfrak{F}_n: n \in N\}$ be a closed network for M such that each $\mathfrak{F}_n$ is discrete. For each $n \in N$, let $\{G_F: F \in \mathfrak{F}_n\}$ be an open discrete family of X such that $F \subset G_F$ for each $F \in \mathfrak{F}_n$ and $\bigcup\{G_F: F \in \mathfrak{F}_n\} \subset G_n$. Let $\mathfrak{U}_n$ be a closure-preserving open neighborhood base for $\bigcup\{F: F \in \mathfrak{F}_n\}$. Set

$$\mathfrak{U}'_n = \left\{U \cap G_F: F \in \mathfrak{F}_n,\ U \in \mathfrak{U}_n,\ \overline{U} \subset \bigcup\ \{G_F: F \in \mathfrak{F}_n\}\right\}.$$

Note that $\mathfrak{U}'_n$ is closure-preserving in X. For the family $\mathfrak{U}$, take the collection consisting of all unions of the sets from $\bigcup\{\mathfrak{U}'_n: n \in N\}$. Then it is easily shown that $\mathfrak{U}$ has the desired property.

(2) $\to$ (3). Let $X' = X \cup F$, where F, X are closed in X'. Suppose $\mathfrak{U} = \{U_\alpha: \alpha \in A\}$ is a closure-preserving open family of the subspace F. By (2) there exists an open family $\mathfrak{V}$ of X such that $\mathfrak{V}$ is closure-preserving in $X - F$ and for each open set W of X there exists $V \in \mathfrak{V}$ such that $V \subset W$ and $V \cap (F \cap X) = W \cap (F \cap X)$. For each $\alpha \in A$, we construct $\mathfrak{V}_\alpha$ as follows:

$$\mathfrak{V}_\alpha = \{V \in \mathfrak{V}: V \cap (F \cap X) = U_\alpha \cap (F \cap X),\ V \subset (U_\alpha \cap F \cap X)'\},$$

where $(U_\alpha \cap F \cap X)'$ is the special extension of $U_\alpha \cap F \cap X$ to X assured by Lemma 1. Write $\mathfrak{V}_\alpha = \{V_\beta: \beta \in B_\alpha\}$, $\alpha \in A$. Set for each $\alpha \in A$, $U'_\beta = V_\beta \cup U_\alpha$, $\beta \in B_\alpha$. Then it is easily seen that $\{U'_\beta: \beta \in B_\alpha,\ \alpha \in A\}$ satisfies the required conditons. This completes the proof.

COROLLARY 1. *Let $X \in \mathfrak{P}$ and let M be a closed subspace of X. If $\mathfrak{U}$ is a closure-preserving open family of M, then there exists a closure-preserving family $\mathfrak{U}'$ of open sets of X such that for every open subset V of X with $V \cap M \in \mathfrak{U}$, there exists $U' \in \mathfrak{U}'$ such that $U' \subset V$ and $U' \cap M = V \cap M$.*

The following is easily obtained by a repetition of the proof of [**10**, Theorem 1]:

COROLLARY 2. *Let $\{X_i: i \in N\}$ be a closed cover of a space X. If each $X_i \in \mathfrak{P}$ and $\{X_i\}$ dominates X, then $X \in \mathfrak{P}$.*

The following Corollary 3 and Theorem 2 are obtained by modifying the proof [**10**, Lemma 2]. We give the proof only to Theorem 2, and that of Corollary 3 is similar.

COROLLARY 3. *The adjunction space of X, $Y \in \mathcal{P}$ belongs to $\mathcal{P}$.*

THEOREM 2. *Let $X \in \mathcal{P}$ and let Y be an M_1-space. Then the adjunction space $Z = X \cup_f Y$ is an M_1-space.*

PROOF. Let f be a mapping of a closed subset H of X into Y. Let $p\colon X \vee Y \to Z$ be the quotient mapping. Let $\mathcal{U} = \bigcup\{\mathcal{U}_n\colon n \in N\}$ be a σ-closure-preserving base for $p(Y)$, where each $\mathcal{U}_n = \{U_\alpha\colon \alpha \in A_n\}$ is closure-preserving in $p(Y)$. By (2) of Theorem 1, there exists an open family $\mathcal{V}$ of X such that V is closure-preserving in $X - H$ and for each open set U of X there exists $V \in \mathcal{V}$ such that $V \subset U$ and $V \cap H = U \cap H$. Let $n \in N$ be fixed for a while. Set

$$\mathcal{V}(\alpha) = \left\{V \in \mathcal{V}\colon V \cap H = p_X^{-1}(U_\alpha),\ V \subset \left(p_X^{-1}(U_\alpha)\right)'\right\}, \qquad \alpha \in A_n,$$
$$\mathcal{U}(\alpha) = \{p(V) \cup U_\alpha\colon V \in \mathcal{V}(\alpha)\}, \qquad \alpha \in A_n,$$

where $p_X = p \mid X$ and $(p_X^{-1}(U_\alpha))'$ is the special extension of $p_X^{-1}(U_\alpha)$ to X assured by Lemma 1. Set $\mathcal{W} = \bigcup\{\mathcal{U}(\alpha)\colon \alpha \in A_n,\ n \in N\}$. Then it is easily seen that $\mathcal{W}$ is a σ-closure-preserving open family of Z, which forms a local base of each point of $p(Y)$ in Z. Since there exists a σ-closure-preserving open family of Z which forms a local base of each point of $Z - p(Y)$ in Z, Z is shown to be an M_1-space.

LEMMA 2. *Let $Z = X \cup Y$ be a stratifiable space, where $X \cap Y = \varnothing$, X is closed in Z and Y is σ-discrete. If $X \in \mathcal{P}$, then $Z \in \mathcal{P}$.*

PROOF. The proof is due to Gruenhage [**5**, Lemma 6.5]. Let $Y = \bigcup\{F_n\colon n \in N\}$, where each F_n is closed discrete in Y and $F_m \cap F_n = \varnothing$ if $m \neq n$. For each $x \in Y$ let $n(x)$ be the integer such that $x \in F_{n(x)}$. Let H be an arbitrary closed subset of Z. Since $X \in \mathcal{P}$, $X \cap H$ has a closure-preserving open neighborhood base $\mathcal{U}$. Take $U \in \mathcal{U}$. Since $Y \cup U$ is stratifiable, the closed subset $H \cup U$ has a closure-preserving neighborhood base $\mathcal{B}$ in $Y \cup U$, which consists of closed neighborhoods in $Y \cup U$. Inductively, define, for each $x \in Y$, an open neighborhood $U(x)$ of x in Y such that

(i) $\{U(x)\colon x \in F_n\}$ is a discrete, closed and open family of Y,

(ii)

$$U(x) \subset D\big(\{x\}, \bigcup\{F_i\colon i < n(x)\} \cup \big(\bigcup\{B \in \mathcal{B}\colon x \notin B\}\big)\big)$$
$$\cap \bigcap\{U(y)\colon x \in U(y),\ n(y) < n(x)\}.$$

Then $\{U(x)\colon x \in Y\}$ has the following properties:

(1) $y \in U(x)$ implies $U(y) \subset U(x)$,

(2) if H is closed in Y, then $\bigcup\{U(y)\colon y \in H\}$ is closed and open in Y, and

(3) $x \notin B \in \mathcal{B}$ implies $U(x) \cap B = \varnothing$.

For every open set V of Z satisfying $V \cap X = U$ and $H \subset V \subset U'$, where U' is the special extension of an open set $H \cup U$ in the subspace $H \cup X$ to Z, define $W(V) = V - \bigcup\{U(x)\colon x \in Y - V\}$. Set $\mathcal{W}(U) = \{W(V)\colon V$ are the above open

sets}, $\mathfrak{W} = \bigcup\{\mathfrak{W}(U): U \in \mathfrak{U}\}$. Then it is easily seen that $\mathfrak{W}$ is a closure-preserving open neighborhood base of H in Z.

THEOREM 3. *If X is a space such that each closed subspace of X belongs to $\mathfrak{P}$, then the closed image of X belongs to $\mathfrak{P}$.*

PROOF. Let $f: X \to Y$ be a closed mapping. By [**5**, Lemma 6.1] there exists a closed subset X_0 of X such that $f \mid X_0: X_0 \to f(X_0)$ is irreducible and $Y - f(X_0)$ is open and σ-discrete. Since $f(X_0) \in \mathfrak{P}$ by [**2**, Lemma 3.3], we have by the above lemma $Y \in \mathfrak{P}$.

THEOREM 4. *Let X be a space such that, for each metrizable space Y, every closed subspace of $X \times Y$ is an M_1-space. Then $X \in \mathfrak{P}$.*

PROOF. Let F be a closed subset of X. Since X is perfectly normal and submetrizable by the assumption, there exists a contraction ρ of X onto a metric space $\hat{X}$ such that $\rho(F)$ is closed in $\hat{X}$. There exists a perfect mapping g of a zero-dimensional metric space Y onto $\hat{X}$. Let Z' be the subspace of $X \times Y$ defined by

$$Z' = \{(x, y) \in X \times Y: \rho(x) = g(y)\}.$$

Let f', σ be the restrictions to Z' of the projections onto X, Y, respectively. By the argument of [**11**, Lemma 5.13, p. 293] f' is a perfect mapping of Z' onto X and σ is a contraction onto Y. By [**11**, Proposition 2.5, p. 219] there exists a closed subspace Z of Z' such that $f = f' \mid Z$ is a perfect and irreducible mapping of Z onto X. Observe that $\sigma(f'^{-1}(F))$ is closed in Y. Therefore $f^{-1}(F)$ has the form

$$f^{-1}(F) = \bigcap\{G_n: n \in N\}, \qquad G_1 = Z, G_{n+1} \subset G_n, n \in N,$$

where each G_n is a closed and open set of Z. By the assumption, there exists a base $\mathfrak{B} = \bigcup\{\mathfrak{B}_i: i \in N\}$ for Z, where each $\mathfrak{B}_i$ is closure-preserving in Z. Set $B_i = B \cap G_i$, $B \in \mathfrak{B}_i$, $i \in N$. Let $\{\mathfrak{S}_\beta: \beta \in \Gamma\}$ be the totality of subcollections of $\mathfrak{B}$. Set

$$V_{\beta i} = \bigcup\{B_i: B \in \mathfrak{S}_\beta \cap \mathfrak{B}_i\}, \quad V_\beta = \bigcup\{V_{\beta i}: i \in N\},$$

$$B_0 = \{\beta \in \Gamma: V_\beta \text{ is an open set such that } f^{-1}(F) \subset V_\beta\},$$

$$U_\beta = X - f(Z - V_\beta), \qquad \beta \in B_0,$$

$$\mathfrak{U} = \{U_\beta: \beta \in B_0\}.$$

By a routine check, it is shown that $\{V_\beta: \beta \in B_0\}$ is a closure-preserving open neighborhood base of $f^{-1}(F)$ in Z. Since f is closed and irreducible, $\mathfrak{U}$ is a closure-preserving open neighborhood base of F in X by [**2**, Lemma 3.3]. This completes the proof.

The author does not know the following: (1) Does every closed subspace of a space $\in \mathfrak{P}$ belong to $\mathfrak{P}$? (2) If $X, Y \in \mathfrak{P}$, does $X \times Y \in \mathfrak{P}$?

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