ON A CERTAIN CLASS OF $M_1$-SPACES

T. MIZOKAMI

Abstract. Let $\mathcal{M}$ be the class of all $M_1$-spaces whose every closed subset has a closure-preserving open neighborhood base. A characterization is given, and it is proved that the adjunction space $X \cup_f Y$ is an $M_1$-space if $X \in \mathcal{M}$ and $Y$ is an $M_1$-space. Moreover, it is proved that if $X$ is a space such that for each metrizable space $Y$, every closed subspace of $X \times Y$ is an $M_1$-space, then $X \in \mathcal{M}$.

1. Introduction. Let $(P_i)$, $i = 1, 2, 3, 4$, be the following statements concerning $M_1$-spaces [3] and stratifiable spaces [1]:

$(P_1)$ Every stratifiable space is an $M_1$-space.

$(P_2)$ Each closed subspace of an $M_1$-space is also an $M_1$-space.

$(P_3)$ Each adjunction space of $M_1$-spaces is also an $M_1$-space.

$(P_4)$ Each closed subset of an $M_1$-space has a $\sigma$-closure-preserving open neighborhood base.

The problems whether the $(P_i)$ are true or not are still open. The problems of $(P_1)$, $(P_2)$ and $(P_3)$ are posed by Ceder [3]. On the other hand, the problem of $(P_4)$ is posed by Borges and Lutzer in [2], where they characterized stratifiable spaces (which coincide with $M_2$-spaces by [4 and 8]) as paracompact $\sigma$-spaces such that each closed subset has a $\sigma$-closure-preserving neighborhood base which consists of not necessarily open neighborhoods. By Borges [1] and by the recent result of Heath and Junnila [6], it follows that $(P_1)$ $\Rightarrow$ $(P_2)$ $\Rightarrow$ $(P_3)$ $\Rightarrow$ $(P_4)$. As is well known by Ceder [3, Lemma 7.3], every closed subset of an $M_2$-space has a closure-preserving neighborhood base. Corresponding to this, let $(P_4)'$ be the statement as follows:

$(P_4)'$ Every closed subset of an $M_1$-space has a closure-preserving open neighborhood base.

Obviously $(P_4)' \Rightarrow (P_4)$. The author in this paper studies the class $\mathcal{M}$ of $M_1$-spaces satisfying $(P_4)'$ and proves that if $(P_2)$ is assumed to be true, then $(P_4)'$ holds.

In this paper, all spaces are assumed to be regular and Hausdorff and all mappings to be continuous and onto unless the contrary is stated explicitly. $N$ always denotes the positive integers. To state the results and the proofs, we give the definitions to the terminology used below. A mapping $f : X \to Y$ is said to be irreducible if no closed proper subspace of $X$ is mapped onto $Y$ by $f$. A space $X$ is said to be monotonically normal [7] if the following is satisfied: To each pair $(H, K)$
of separated subsets of $X$ one can assign an open set $D(H, K)$ in such a way that

(i) $H \subset D(H, K) \subset D(H, K) \subset X - K$, and

(ii) if $(H', K')$ is a pair of separated sets having $H \subset H'$ and $K' \subset K$, then $D(H, K) \subset D(H', K')$.

According to [7], every stratifiable space is monotonically normal, and this fact is used below.

2. The class $\mathcal{P}$.

Definition 1. Let $\mathcal{P}$ be the class of all $M_1$-spaces satisfying $(P_4)'$.

Most common examples of $M_1$-spaces seem to belong to $\mathcal{P}$. For example: (1) stratifiable $F_\sigma$-metrizable spaces due to Gruenhage [5], (2) $M_1$-spaces with dim $\leq 0$ and (3) the closed image of an $M_0$-space (a space which has a $\sigma$-closure-preserving base consisting of closed and open sets) belong to $\mathcal{P}$. The proofs are given below implicitly. Every Lašnev space, and more generally every $L$-space of Nagami [9], belongs to $\mathcal{P}$, but the converse is not true as shown by [9, Examples 2.1, 2.2].

Lemma 1. Let $M$ be a closed subspace of a stratifiable space $X$. Then to each open set $U$ of $M$, we can assign an open set $U'$ of $X$ such that $U' \cap M = U$ and the following is satisfied:

$\ast$ $U' \cap V' = \emptyset$ whenever $U, V$ are disjoint open sets of $M$.

Proof. Let $\mathcal{F}_n = \bigcup \{ F^n_n : n \in \mathbb{N} \}$ be a $\sigma$-locally finite closed network of $M$ such that each $F^n_n$ is locally finite and $F^n_n \subset F^{n+1}_n$. For each open set $W$ of $M$ and each $n \in \mathbb{N}$, set

$$
F^n_n(W) = \{ F \in F^n_n : F \subset W \}, \quad F^n_n(W)^c = \bigcup \{ F : F \in F^n_n(W) \}.
$$

We shall construct $U'$ for an arbitrary open set $U$ of $M$ by

$$
U_n = D\left(F^n_n(U)^c, M - U\right) - D\left(F^n_n(M - U)^c, U\right), \quad n \in \mathbb{N},
$$

$$
U' = \bigcup \{ U_n : n \in \mathbb{N} \}.
$$

Obviously $U' \cap M = U$. To see $(\ast)$ for disjoint open sets $U, V$ of $M$, assume that $V \cap U_n \neq \emptyset, m, n \in \mathbb{N}$. Without loss of generality, we can assume $n \geq m$. Observe

$$
D\left(F^m_m(V)^c, M - V\right) \subset D\left(F^n_n(M - U)^c, U\right).
$$

This contradicts the fact $V \cap U_n \neq \emptyset$.

In the above proof, if we use only the fact that $X$ is monotonically normal, we obtain the following:

Corollary. Let $M$ be a closed subspace of a monotonically normal space $X$ and let $\{ U_{\alpha} : \alpha \in A \}$ be an open family of $M$. Then there exists an open family $\{ U'_{\alpha} : \alpha \in A \}$ of $X$ such that $U_{\alpha} \cap M = U'_{\alpha}$ for each $\alpha \in A$ and for each $B \subset A$

$$
\bigcup \{ U_{\alpha} : \alpha \in B \} = \bigcup \{ U'_{\alpha} : \alpha \in B \} \cap M.
$$

In fact, if we define

$$
U_{\alpha} = \bigcup \{ D\{x\}, M - U_{\alpha} : x \in U_{\alpha} \}, \quad \alpha \in A,
$$

then $\{ U'_{\alpha} : \alpha \in A \}$ satisfied the required property.
Definition 2. (Nagata [10, Definition 2]). A space \( X \) is said to have the property (ECP) when the following is satisfied: If \( X' = X \cup F \), where \( X, F \) are closed in \( X' \), and if \( \mathcal{U} = \{ U_\alpha : \alpha \in A \} \) is a closure-preserving open collection in the subspace \( F \), then for each \( \alpha \in A \) there is a family \( \{ U_\beta : \beta \in B_\alpha \} \) of open sets of \( X' \) satisfying \( \mathcal{U}' = \{ U_\beta' : \beta \in \bigcup \{ B_\alpha : \alpha \in A \} \} \) is closure-preserving in \( X' \), (2) for each \( \beta \in B_\alpha \), \( U_\beta \cap F = U_\alpha \) and for every open set \( V \) in \( X' \) with \( V \cap F = U_\alpha \), there is \( \beta \in B_\alpha \) such that \( U_\beta \subset U_\beta' \subset V \), (3) for every open set \( W \) of \( F \), there is an open set \( W' \) of \( X' \) such that \( W' \cap F = W \) and such that \( W' \cap U_\beta' = \emptyset \) whenever \( \beta \in B_\alpha \) and \( W \cap U_\alpha = \emptyset \).

Theorem 1. The following are equivalent for a stratifiable space \( X \):

(1) \( X \in \mathcal{P} \).

(2) For each closed subspace \( M \) of \( X \), there exists an open family \( \mathcal{U} \) of \( X \) such that \( \mathcal{U} \) is closure-preserving in \( X - M \) and for each open set \( V \) of \( X \) there exists \( U \in \mathcal{U} \) such that \( U \subset V \) and \( U \cap M = V \cap M \).

(3) \( X \) has (ECP).

Proof. (3) \( \rightarrow \) (1) is trivial.

(1) \( \rightarrow \) (2): Let \( \{ G_n : n \in \mathbb{N} \} \) be a sequence of open sets of \( X \) such that \( M = \bigcap \{ G_n : n \in \mathbb{N} \} \), \( G_{n+1} \subset G_n \), \( n \in \mathbb{N} \). Let \( \mathcal{G} = \bigcup \{ \mathcal{G}_n : n \in \mathbb{N} \} \) be a closed network for \( M \) such that each \( \mathcal{G}_n \) is discrete. For each \( n \in \mathbb{N} \), let \( \{ G_F : F \in \mathcal{G}_n \} \) be an open discrete family of \( X \) such that \( F \subset G_F \) for each \( F \in \mathcal{G}_n \) and \( \bigcup \{ G_F : F \in \mathcal{G}_n \} \subset G_n \). Let \( \mathcal{U}_n \) be a closure-preserving open neighborhood base for \( \bigcup \{ F : F \in \mathcal{G}_n \} \). Set \( \mathcal{U}_n' = \{ U \cap G_F : F \in \mathcal{G}_n, U \in \mathcal{U}_n, \bar{U} \subset \bigcup \{ G_F : F \in \mathcal{G}_n \} \} \).

Note that \( \mathcal{U}_n' \) is closure-preserving in \( X \). For the family \( \mathcal{U} \), take the collection consisting of all unions of the sets from \( \bigcup \{ \mathcal{U}_n' : n \in \mathbb{N} \} \). Then it is easily shown that \( \mathcal{U} \) has the desired property.

(2) \( \rightarrow \) (3). Let \( X' = X \cup F \), where \( F, X \) are closed in \( X' \). Suppose \( \mathcal{U} = \{ U_\alpha : \alpha \in A \} \) is a closure-preserving open family of the subspace \( F \). By (2) there exists an open family \( \mathcal{V} \) of \( X \) such that \( \mathcal{V} \) is closure-preserving in \( X - F \) and for each open set \( W \) of \( X \) there exists \( V \in \mathcal{V} \) such that \( V \subset W \) and \( V \cap (F \cap X) = W \cap (F \cap X) \).

For each \( \alpha \in A \), we construct \( \mathcal{V}_\alpha \) as follows:

\[
\mathcal{V}_\alpha = \left\{ V \in \mathcal{V} : V \cap (F \cap X) = U_\alpha \cap (F \cap X), V \subset (U_\alpha \cap F \cap X)' \right\},
\]

where \( (U_\alpha \cap F \cap X)' \) is the special extension of \( U_\alpha \cap F \cap X \) to \( X \) assured by Lemma 1. Write \( \mathcal{V}_\alpha = \{ V_\beta : \beta \in B_\alpha \}, \alpha \in A \). Set for each \( \alpha \in A \), \( U_\alpha' = V_\beta \cup U_\alpha, \beta \in B_\alpha \). Then it is easily seen that \( \{ U_\beta' : \beta \in B_\alpha, \alpha \in A \} \) satisfies the required conditions. This completes the proof.

Corollary 1. Let \( X \in \mathcal{P} \) and let \( M \) be a closed subspace of \( X \). If \( \mathcal{U} \) is a closure-preserving open family of \( M \), then there exists a closure-preserving family \( \mathcal{U}' \) of open sets of \( X \) such that for every open subset \( V \) of \( X \) with \( V \cap M \in \mathcal{U} \), there exists \( U' \in \mathcal{U}' \) such that \( U' \subset V \) and \( U' \cap M = V \cap M \).

The following is easily obtained by a repetition of the proof of [10, Theorem 1]:

Corollary 2. Let \( \{ X_i : i \in \mathbb{N} \} \) be a closed cover of a space \( X \). If each \( X_i \in \mathcal{P} \) and \( \{ X_i \} \) dominates \( X \), then \( X \in \mathcal{P} \).
The following Corollary 3 and Theorem 2 are obtained by modifying the proof [10, Lemma 2]. We give the proof only to Theorem 2, and that of Corollary 3 is similar.

**Corollary 3.** The adjunction space of $X, Y \in \mathcal{P}$ belongs to $\mathcal{P}$.

**Theorem 2.** Let $X \in \mathcal{P}$ and let $Y$ be an $M_1$-space. Then the adjunction space $Z = X \cup Y$ is an $M_1$-space.

**Proof.** Let $f$ be a mapping of a closed subset $H$ of $X$ into $Y$. Let $p: X \vee Y \to Z$ be the quotient mapping. Let $\mathfrak{U} = \bigcup \{\mathfrak{B}_n: n \in \mathbb{N}\}$ be a $\sigma$-closure-preserving base for $p(Y)$, where each $\mathfrak{B}_n = \{U_\alpha: \alpha \in A_n\}$ is closure-preserving in $p(Y)$. By (2) of Theorem 1, there exists an open family $\mathcal{V}$ of $X$ such that $V$ is closure-preserving in $X - H$ and for each open set $U$ of $X$ there exists $V \in \mathcal{V}$ such that $V \subseteq U$ and $V \cap H = U \cap H$. Let $n \in \mathbb{N}$ be fixed for a while. Set

$$\mathcal{V}(\alpha) = \{V \in \mathcal{V}: V \cap H = p^{-1}_X(U_\alpha), V \subseteq (p^{-1}_X(U_\alpha))^\prime\}, \quad \alpha \in A_n,$$

$$\mathfrak{U}(\alpha) = \{p(V) \cup U_\alpha: V \in \mathcal{V}(\alpha)\}, \quad \alpha \in A_n,$$

where $p_X = p|X$ and $(p^{-1}_X(U_\alpha))^\prime$ is the special extension of $p^{-1}_X(U_\alpha)$ to $X$ assured by Lemma 1. Set $\mathcal{W} = \bigcup \{\mathfrak{U}(\alpha): \alpha \in A_n, n \in \mathbb{N}\}$. Then it is easily seen that $\mathcal{W}$ is a $\sigma$-closure-preserving open family of $Z$, which forms a local base of each point of $p(Y)$ in $Z$. Since there exists a $\sigma$-closure-preserving open family of $Z$ which forms a local base of each point of $Z - p(Y)$ in $Z$, $Z$ is shown to be an $M_1$-space.

**Lemma 2.** Let $Z = X \cup Y$ be a stratifiable space, where $X \cap Y = \emptyset$, $X$ is closed in $Z$ and $Y$ is $\sigma$-discrete. If $X \in \mathcal{P}$, then $Z \in \mathcal{P}$.

**Proof.** The proof is due to Gruenhage [5, Lemma 6.5]. Let $Y = \bigcup \{F_n: n \in \mathbb{N}\}$, where each $F_n$ is closed discrete in $Y$ and $F_m \cap F_n = \emptyset$ if $m \neq n$. For each $x \in Y$ let $n(x)$ be the integer such that $x \in F_{n(x)}$. Let $H$ be an arbitrary closed subset of $Z$. Since $X \in \mathcal{P}$, $X \cap H$ has a closure-preserving open neighborhood base $\mathfrak{U}$. Take $U \in \mathfrak{U}$. Since $Y \cup U$ is stratifiable, the closed subset $H \cup U$ has a closure-preserving neighborhood base $\mathfrak{B}$ in $Y \cup U$, which consists of closed neighborhoods in $Y \cup U$. Inductively, define, for each $x \in Y$, an open neighborhood $U(x)$ of $x$ in $Y$ such that

(i) $\{U(x): x \in F_n\}$ is a discrete, closed and open family of $Y$,

(ii) $U(x) \subseteq D(\{x\}, \bigcup \{F_i: i < n(x)\} \cup \bigcup \{B \in \mathfrak{B}: x \notin B\}.)$

$$\cap \cap \{U(y): x \in U(y), n(y) < n(x)\}.$$

Then $\{U(x): x \in Y\}$ has the following properties:

(1) $y \in U(x)$ implies $U(y) \subseteq U(x)$,

(2) if $H$ is closed in $Y$, then $\bigcup \{U(y): y \in H\}$ is closed and open in $Y$, and

(3) $x \notin B \in \mathfrak{B}$ implies $U(x) \cap B = \emptyset$.

For every open set $V$ of $Z$ satisfying $V \cap X = U$ and $H \subseteq V \subseteq U'$, where $U'$ is the special extension of an open set $H \cup U$ in the subspace $H \cup X$ to $Z$, define $W(V) = V - \bigcup \{U(x): x \in Y - V\}$. Set $\mathfrak{W}(U) = \{W(V): U$ are the above open
sets), \( \mathcal{W} = \bigcup \{ \mathcal{M}(U) : U \in \mathcal{I} \} \). Then it is easily seen that \( \mathcal{W} \) is a closure-preserving open neighborhood base of \( H \) in \( Z \).

**Theorem 3.** If \( X \) is a space such that each closed subspace of \( X \) belongs to \( \mathcal{V} \), then the closed image of \( X \) belongs to \( \mathcal{V} \).

**Proof.** Let \( f : X \to Y \) be a closed mapping. By [5, Lemma 6.1] there exists a closed subset \( X_0 \) of \( X \) such that \( f|_{X_0} : X_0 \to Y \) is irreducible and \( Y - f(X_0) \) is open and \( \sigma \)-discrete. Since \( f(X_0) \in \mathcal{V} \) by [2, Lemma 3.3], we have by the above lemma \( Y \in \mathcal{V} \).

**Theorem 4.** Let \( X \) be a space such that, for each metrizable space \( Y \), every closed subspace of \( X \times Y \) is an \( M_1 \)-space. Then \( X \in \mathcal{V} \).

**Proof.** Let \( F \) be a closed subset of \( X \). Since \( X \) is perfectly normal and submetrizable by the assumption, there exists a contraction \( \rho \) of \( X \) onto a metric space \( \hat{X} \) such that \( \rho(F) \) is closed in \( \hat{X} \). There exists a perfect mapping \( g \) of a zero-dimensional metric space \( Y \) onto \( \hat{X} \). Let \( Z' \) be the subspace of \( X \times Y \) defined by

\[ Z' = \{ (x, y) \in X \times Y : \rho(x) = g(y) \}. \]

Let \( f' \), \( \sigma \) be the restrictions to \( Z' \) of the projections onto \( X \), \( Y \), respectively. By the argument of [11, Lemma 5.13, p. 293] \( f' \) is a perfect mapping of \( Z' \) onto \( X \) and \( \sigma \) is a contraction onto \( Y \). By [11, Proposition 2.5, p. 219] there exists a closed subspace \( Z \) of \( Z' \) such that \( f = f'|_Z \) is a perfect and irreducible mapping of \( Z \) onto \( X \). Observe that \( \sigma(f^{-1}(F)) \) is closed in \( Y \). Therefore \( f^{-1}(F) \) has the form

\[ f^{-1}(F) = \bigcap \{ G_n : n \in \mathbb{N} \}, \quad G_1 = Z, \ G_{n+1} \subset G_n, n \in \mathbb{N}, \]

where each \( G_n \) is a closed and open set of \( Z \). By the assumption, there exists a base \( \mathcal{B} = \bigcup \{ \mathcal{B}_i : i \in \mathbb{N} \} \) for \( Z \), where each \( \mathcal{B}_i \) is closure-preserving in \( Z \). Set \( B_i = B \cap G_i, B \in \mathcal{B}_i, i \in \mathbb{N} \). Let \( \{ \mathcal{B}_i : \beta \in \Gamma \} \) be the totality of subcollections of \( \mathcal{B} \). Set

\[ \mathcal{V}_{B_i} = \bigcup \{ B_i : B \in \mathcal{B}_i \}, \quad \mathcal{V}_\beta = \bigcup \{ \mathcal{V}_{B_i} : i \in \mathbb{N} \}, \]

\[ \mathcal{B}_0 = \{ \beta \in \Gamma : \mathcal{V}_\beta \text{ is an open set such that } f^{-1}(F) \subset \mathcal{V}_\beta \}, \]

\[ U_\beta = X - f(Z - V_\beta), \quad \beta \in \mathcal{B}_0, \]

\[ \mathcal{W} = \{ U_\beta : \beta \in \mathcal{B}_0 \}. \]

By a routine check, it is shown that \( \{ V_\beta : \beta \in \mathcal{B}_0 \} \) is a closure-preserving open neighborhood base of \( f^{-1}(F) \) in \( Z \). Since \( f \) is closed and irreducible, \( \mathcal{W} \) is a closure-preserving open neighborhood base of \( F \) in \( X \) by [2, Lemma 3.3]. This completes the proof.

The author does not know the following: (1) Does every closed subspace of a space \( \in \mathcal{V} \) belong to \( \mathcal{V} \)? (2) If \( X, Y \in \mathcal{V} \), does \( X \times Y \in \mathcal{V} \)?

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**References**


Department of Mathematics, Joetsu University of Education, Joetsu, Niigata 943, Japan