CELL-LIKE DECOMPOSITIONS OF HOMOGENEOUS CONTINUA

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Abstract. Certain decompositions of homogeneous continua are shown to be cell-like. In particular, the aposyndetic decomposition described by F. B. Jones of a homogeneous, decomposable continuum is cell-like, and we prove that any homogeneous decomposable continuum admits a continuous decomposition into mutually homeomorphic, indecomposable, homogeneous, cell-like terminal continua so that the quotient space is an aposyndetic homogeneous continuum.

D. C. Wilson [8] has shown that a monotone, completely regular map $f : X \to Z$ of the $n$-dimensional continuum $X$ onto the nondegenerate continuum $Y$ has the property that $H^n(f^{-1}(z)) = 0$, for all $z$ in $Z$. In particular, if $n = 1$, then the point inverses of $f$ are acyclic continua.

More recently, Mason and Wilson [5] have shown that if $n = 1$, then the point inverses of $f$ are tree-like continua, that is, $f$ is a cell-like map.

Since the projection maps of a product space onto the factors are completely regular maps, one cannot, in general, extend the Mason-Wilson result to higher dimensions.

The impetus for the Mason-Wilson result, however, was the author's applications [6, 7] of completely regular maps to certain monotone decompositions of homogeneous continua. In this paper, we show that the completely regular, monotone maps arising as quotient maps of these decompositions are cell-like maps. In particular, Jones' aposyndetic decomposition of a homogeneous, decomposable continuum is a decomposition of that continuum into cell-like sets.

A continuum $X$ is cell-like if each mapping of $X$ into a compact ANR is inessential. If the map $f$ is inessential, we write $f \simeq 0$. A continuum is cell-like if and only if it has trivial shape.

A map is cell-like if each of its point inverses is cell-like.

The following theorem is classical.

Wirecutting Theorem. Let $A$ and $B$ be closed subsets of the compact space $M$. If no connected subset of $M$ intersects both $A$ and $B$, then there exist disjoint closed subsets $M_1$ and $M_2$ of $M$ such that $A \subset M_1$, $B \subset M_2$, and $M = M_1 \cup M_2$.

A subcontinuum $Z$ of the continuum $X$ is said to be terminal if each subcontinuum $Y$ of $X$ such that $Y \cap Z \neq \emptyset$ satisfies either $Y \subset Z$ or $Z \subset Y$.

The proof of the next theorem is similar to that of [2, Theorem 2].
THEOREM 1. If $A$ is a terminal subcontinuum of the continuum $X$, if $B$ is a subcontinuum of $X$ disjoint from $A$, and if $f: A \to Y$ is a map of $A$ into the ANR $Y$, then there exists a map $F: X \to Y$ such that $F|A = f$ and $F|B \simeq 0$.

PROOF. There exists an open set $U$ in $X - B$ containing $A$ and a map $g: U \to Y$ extending $f$. There exists an open neighborhood $V$ of a point $a$ in $A$ such that $V \subset U$ and $g|V \simeq 0$, since $Y$ is locally contractible. Since $A - V$ and $X - U$ are closed subsets of $X - V$ such that no connected subset of $X - V$ meets both $A - V$ and $X - U$, the Wirecutting Theorem implies that $X - V$ is the union of the disjoint closed sets $X_1$ and $X_2$, with $A - V \subset X_1$ and $X - U \subset X_2$.

Thus $X_1 \cap A$ and $X_2$ are disjoint closed subsets of $X$. Let $h: X \to I$ be a Urysohn map with $h(X_1 \cup A) = 0$ and $h(X_2) = 1$. Let $M = h^{-1}([0, \frac{1}{2}])$ and $N = h^{-1}([\frac{1}{2}, 1])$. Then $X = M \cup N$, $A \subset M$, $X - U \subset N$, and $M \cap N \subset V - A$. Since the map $g|M \cap N$ is homotopic to a constant map, there exists an inessential map $k: N \to Y$ such that $k|M \cap N = g|M \cap N$.

The map $F: X \to Y$ defined by

$$F(x) = \begin{cases} g(x), & x \in M, \\ k(x), & x \in N, \end{cases}$$

is the desired extension of $f$.

A map $g: X \to Z$ between continua is completely regular if for each $\delta > 0$ and each point $z$ in $Z$, there exists an open set $V$ in $Z$ containing $z$ such that if $z' \in V$ then there is a homeomorphism $h$ from $g^{-1}(z)$ to $g^{-1}(z')$ such that $d(x, h(x)) < \delta$, for each $x$ in $g^{-1}(z)$. Each completely regular map is open.

THEOREM 2. Let $g: X \to Z$ be a monotone, completely regular map of the continuum $X$ onto the nondegenerate continuum $Z$. Let $z_1$ be a point of $Z$. If $g^{-1}(z_1)$ is a terminal subcontinuum of $X$, then $g$ is a cell-like map.

PROOF. Let $Y$ be a compact ANR, and let $f: g^{-1}(z_1) \to Y$ be a map. Let $z_2$ be another point of $Z$, and let $F: X \to Y$ be an extension of $f$ such that $F|g^{-1}(z_2) \simeq 0$.

Since $Y$ is a compact ANR, there exists $\epsilon > 0$ such that for any space $W$ and for any two maps $\alpha, \beta: W \to Y$, $d(\alpha, \beta) < \epsilon$ implies $\alpha \simeq \beta$. Let $\delta$ be a positive number such that $d(x, x') < \delta$ implies $d(F(x), F(x')) < \epsilon$.

Let $Z_2 = \{z \in Z: F|g^{-1}(z) \simeq 0\}$. We show that $Z_2$ is open. Let $z \in Z_2$. Since $g$ is completely regular, there exists an open set $V$ in $Z$ containing $z$ such that if $z' \in V$, then there is a homeomorphism $h$ from $g^{-1}(z)$ to $g^{-1}(z')$ such that $d(x, h(x)) < \delta$, for each $x$ in $g^{-1}(z)$. Hence $d(F|g^{-1}(z'), F|g^{-1}(z) \circ h^{-1}) < \epsilon$, and so $F|g^{-1}(z') \simeq F|g^{-1}(z) \circ h^{-1}$. But the latter map is inessential, since $z \in Z_2$, and so $z' \in Z_2$. Hence $Z_2$ is open. A similar proof shows that $Z_2$ is closed.

Since $Z_2$ is a nonempty open and closed subset of the connected set $Z$, it follows that $Z_2 = Z$ and hence $f \simeq 0$. Thus each mapping of $g^{-1}(z_1)$ into a compact ANR is inessential, and hence $g^{-1}(z_1)$ is cell-like. Since point inverses of completely regular maps defined on continua are homeomorphic, this implies that $g$ is a cell-like map.

The homeomorphism group $H$ of a continuum $X$ is said to respect the decomposition $\mathcal{G}$ of $X$ if $G \in \mathcal{G}$ implies $h(G) \in \mathcal{G}$, for each homeomorphism $h \in H$.

The following theorem is due to E. Dyer (see [3] for a simple proof).

THEOREM 3. Let $X$ and $Y$ be nondegenerate metric continua and let $f: X \to Y$ be a monotone open surjection. Then there exists a dense $G_\delta$-subset $A$ of $Y$ having
the following property: for each \( y \in A \), for each continuum \( B \subset f^{-1}(y) \), for each \( x \) from the interior of \( B \) in \( f^{-1}(y) \) and for each neighborhood \( U \) of \( B \) in \( X \), there exists a continuum \( Z \subset X \) containing \( B \) and a neighborhood \( V \) of \( y \) in \( Y \) such that \( x \in Z^0 \), \((f|Z)^{-1}(V) \subset U \) and \( f|Z : Z \to Y \) is a monotone surjection.

**THEOREM 4.** Let \( X \) be a homogeneous continuum. Let \( \mathcal{G} \) be a partition of \( X \) into terminal subcontinua such that the homeomorphism group of \( X \) respects \( \mathcal{G} \). Then

1. the partition \( \mathcal{G} \) is a continuous decomposition of \( X \).
2. The quotient map \( \pi : X \to Z \) of \( X \) onto the quotient space \( Z \) is completely regular.
3. \( Z \) is a homogeneous continuum.
4. The elements of \( \mathcal{G} \) are mutually homeomorphic, indecomposable, homogeneous, cell-like continua.

**PROOF.** Parts (1), (2), and (3) have been proved in [7, Theorem 4]. Since the homeomorphism group of \( X \) respects \( \mathcal{G} \), it follows that the elements of \( \mathcal{G} \) are mutually homeomorphic and homogeneous. Theorem 2 implies that the elements of \( \mathcal{G} \) are cell-like. As in [4], Dyer's Theorem and the fact that the elements of \( \mathcal{G} \) are terminal continua imply that the proper subcontinua of one (and hence all) of the elements of \( \mathcal{G} \) have empty interiors in that element. Hence the elements of \( \mathcal{G} \) are indecomposable.

An important application of this theorem is the following improvement of Jones’ Aposyndetic Decomposition Theorem [1]. In the case that \( X \) is a curve, this improvement is already known from results of Rogers [6] and Mason and Wilson [5].

**THEOREM 5.** Suppose \( X \) is a homogeneous, decomposable continuum. Then \( X \) admits a continuous decomposition into mutually homeomorphic, indecomposable, homogeneous, cell-like terminal continua so that the quotient space is an aposyndetic homogeneous continuum.

**PROOF.** Jones [1] has shown that his decomposition of \( X \) consists of terminal continua. Furthermore, the homeomorphism group of \( X \) respects this decomposition.

**Question 1.** Is each homogeneous, indecomposable, cell-like continuum tree-like? **Question 2.** Can the aposyndetic decomposition of Theorem 5 raise dimension?

**REFERENCES**

5. A. Mason and D. C. Wilson, Monotone mappings on n-dimensional continua, preprint.

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