

ON THE FIRST ORDER THEORY OF THE ARITHMETICAL DEGREES

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ABSTRACT. The first order theory of the arithmetical degrees with arithmetical jump is not elementarily equivalent to the first order theory of the Turing degrees with jump.

1. The result and some related problems. The *arithmetical degrees* are equivalence classes with respect to the equivalence relation: $A \equiv_a B$ iff $A \leq_a B$ and $B \leq_a A$, where $A \leq_a B$ means that A is arithmetical in B (i.e. definable from B in an arithmetical way). By \mathcal{D}_a we mean the structure of the arithmetical degrees with the partial ordering induced by \leq_a on the equivalence classes. By \mathcal{D}'_a we mean \mathcal{D}_a with the unary operation (called *arithmetical jump*) induced by the ω -jump (Rogers [5, p. 258]) on the equivalence classes. Some results about the structures \mathcal{D}_a and \mathcal{D}'_a are collected in our paper [4] (to which we refer for notations, unexplained terminology and background). These structures have been less studied than the structures of the Turing degrees with or without jump (\mathcal{D} and \mathcal{D}' , respectively). Harding [2] has been able to show (by using initial segments embeddings, hence by quite sophisticated methods) that \mathcal{D}'_a and \mathcal{D}' are not isomorphic. The purpose of this note is to prove the following:

THEOREM. \mathcal{D}'_a is not elementarily equivalent to \mathcal{D}' .

The proof is simple and does not require new methods beyond those introduced in Sacks [6]. Nevertheless we think the result is a step toward a better understanding of the arithmetical jump, a seemingly refractory object to study. An apparently difficult problem about it is the following:

Problem 1. Determine the range of the arithmetical jump operator restricted to the arithmetical degrees below $\mathbf{0}'$.

The referee has noted that not every arithmetical degree between $\mathbf{0}'$ and $\mathbf{0}''$ is the jump of an arithmetical degree below $\mathbf{0}'$, since if $A \leq_a \mathbf{0}^\omega$ then A^ω is an arithmetical singleton, but there exists B such that $\mathbf{0}^\omega \leq_a B \leq_a \mathbf{0}^{\omega+\omega}$ and B is not an arithmetical singleton. We thus propose the following:

Conjecture. If $\mathbf{a} < \mathbf{0}'$ then $\mathbf{a}' = \mathbf{0}'$.

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This would imply the Theorem of this note, since the analogous fact fails for Turing degrees; see Shoenfield [7].

It may be worth noting that for the structure \mathcal{D}'_h of hyperdegrees with hyperjump (see Rogers [5, Chapter 16]) the analogous fact holds (Spector [9]): this shows $\mathcal{D}'_h \not\equiv \mathcal{D}'_a$. That $\mathcal{D}'_h \not\equiv \mathcal{D}'_a$ follows from the fact that the smallest ideal closed under jump has least upper bound in \mathcal{D}'_h but not in \mathcal{D}'_a .

A solution to the following problem might also shed light on the nature of the arithmetical jump operator.

Problem 2. Are the arithmetical degrees below $\mathbf{0}'$ and the Turing degrees below $\mathbf{0}^\omega$ elementarily equivalent?

Note that the *sets* involved here are respectively those arithmetical and recursive in $\mathbf{0}^\omega$.

A natural question concerns the extension of the elementary inequivalence proved here to the structures without jump; in particular,

Problem 3. Is \mathcal{D}_a elementarily equivalent to \mathcal{D} ?

With respect to the similarly open questions:

Is \mathcal{D}_h elementarily equivalent to \mathcal{D}_a ?

Is \mathcal{D}_h elementarily equivalent to \mathcal{D} ?

It is at least known that the negative answers to them are consistent with ZFC. Indeed Simpson [8] has proved that if $V = L$ then there is no cone of minimal covers of hyperdegrees, while such cones are known to exist for both Turing degrees and arithmetical degrees (by Borel determinacy and a lemma of Martin [3]).

2. The proof. Let I be an ideal: we say that \mathbf{a} is a 1-least upper bound (1-l.u.b.) for I if it is the least element of $\{\mathbf{c}' : (\forall \mathbf{b} \in I)(\mathbf{b} \leq \mathbf{c}')\}$. Sacks [6] has proved that in \mathcal{D}' the smallest jump-ideal has no 1-l.u.b.. It is enough to show that in \mathcal{D}'_a the smallest jump-ideal has 1-l.u.b. (Note that “the smallest jump-ideal has 1-l.u.b.” is indeed expressible as a first order sentence in both \mathcal{D}' and \mathcal{D}'_a , because in both every countable ideal has an exact pair. So quantification over ideals can be replaced by quantification over pairs of degrees. See [4] for details.)

Since some confusion might arise between arithmetical degrees and Turing degrees, we recall (see [4]) some definitions:

$\mathbf{0}^n$ = the arithmetical degree of $0^{\omega \cdot n}$ (remember that the arithmetical jump is induced by the ω -jump operator);

$\mathbf{0}^\omega$ = the arithmetical degree of 0^{ω^2} .

The smallest jump-ideal of \mathcal{D}'_a is $I = \{\mathbf{b} : (\exists n)(\mathbf{b} \leq \mathbf{0}^n)\}$. We show that $\mathbf{0}^\omega = 1$ -l.u.b. of I .

First note that if $(\forall n)(0^{\omega \cdot n} \leq_a A)$ then $0^{\omega^2} \leq_a A^\omega$ (in terms of arithmetical degrees this says: if $(\forall n)(\mathbf{0}^n \leq \mathbf{a})$ then $\mathbf{0}^\omega \leq \mathbf{a}'$). Indeed, if $0^{\omega \cdot n} \leq_a A$ then for some $m, 0^{\omega \cdot n} \leq_T A^m$ and so $0^{\omega \cdot n} \leq_T A^\omega$. By the Enderton-Putnam [1] computation, if this holds for every n then $0^{\omega^2} \leq_T A^{\omega+2}$ and so $0^{\omega^2} \leq_a A^\omega$.

It is now enough to show that there is B such that $(\forall n)(0^{\omega \cdot n} \leq_a B)$ and $B^\omega \leq_a 0^{\omega^2}$ (in terms of arithmetical degrees this says that $(\forall n)(\mathbf{0}^n \leq \mathbf{b})$ and $\mathbf{b}' \leq \mathbf{0}^\omega$). Note that we cannot get B such that $(\forall n)(0^{\omega \cdot n} \leq_T B)$ since otherwise (by the same computation above) $0^{\omega^2} \leq_T B^2$ and so $B^\omega \leq_a B$. We build a sequence $\langle T_n \rangle_{n \in \omega}$ of arithmetically pointed trees (i.e. trees arithmetical in every one of their branches) of

arithmetical degree $\mathbf{0}^n$. Two basic lemmas from Sacks [6] are:

(1) If P is arithmetically pointed, $Q \subseteq P$ and $Q \leq_a P$, then Q is arithmetically pointed and $Q \equiv_a P$.

(2) If P is arithmetically pointed and $P \leq_a A$, then there is an arithmetically pointed $Q \subseteq P$ such that $Q \equiv_a A$.

We will have $B \in \bigcap_{n \in \omega} T_n$ and by pointedness this will ensure $(\forall n)(0^{\omega \cdot n} \leq_a B)$. To get $B^\omega \leq_a 0^{\omega^2}$ let $\langle \varphi_n \rangle_{n \in \omega}$ be a recursive enumeration of the arithmetical sentences $n \in X^\omega$. Let $T_0 =$ full binary tree. Given T_n let $Q_n \subseteq T_n$ be a subtree of it of sufficiently generic sets, such that one of the following holds:

For every A on Q_n , $A \models \varphi_n$.

For every A on Q_n , $A \models \sim \varphi_n$.

Sufficiently generic means m -generic for the least m such that $\varphi_n \in \Sigma_m^0$. Q_n is built by local forcing on T_n and is clearly possible to get it arithmetical in T_n (since φ_n is arithmetical). So by lemma (1), Q_n is arithmetically pointed and $Q_n \equiv_a T_n$. Let $T_{n+1} \subseteq Q_n$ be an arithmetically pointed tree of arithmetical degree $\mathbf{0}^{n+1}$ (by lemma (2)). The construction is clearly recursive in 0^{ω^2} , and so $B^\omega \leq_a 0^{\omega^2}$ (actually $B^\omega \leq_T 0^{\omega^2}$), because to decide if $n \in B^\omega$ it is enough to see whether for any branch A of T_{n+1} , $A \models \varphi_n$. Choose A recursive in T_{n+1} (e.g. the leftmost branch) and see if $A \models \varphi_n$; this is arithmetical in A , hence arithmetical in T_{n+1} and thus recursive in $0^{\omega \cdot (n+2)} \leq_T 0^{\omega^2}$ (uniformly in n).

BIBLIOGRAPHY

1. Enderton and Putnam, *A note on the hyperarithmetical hierarchy*, J. Symbolic Logic **35** (1970), 429–430.
2. Harding, *Forcing in recursion theory*, Ph. D. Thesis, University College of Swansea, 1974.
3. Martin, *The axiom of determinateness and reduction principles in the arithmetical hierarchy*, Bull. Amer. Math. Soc. **74** (1968), 687–689.
4. P. Odifreddi, *Forcing and reducibilities*, J. Symbolic Logic **47** (1982).
5. Rogers, *Theory of recursive functions and effective computability*, McGraw-Hill, New York, 1967.
6. G. Sacks, *Forcing with perfect closed sets*, Proc. Sympos. Pure Math., vol. 13, Amer. Math. Soc., Providence, R. I., 1971, pp. 331–355.
7. J. Shoenfield, *On degrees of unsolvability*, Ann. of Math. (2) **69** (1959), 644–653.
8. Simpson, *Minimal covers and hyperdegrees*, Trans. Amer. Math. Soc. **209** (1975), 45–64.
9. Spector, *Recursive well-orderings*, J. Symbolic Logic **20** (1955), 151–163.

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