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EMBEDDINGS IN MINIMAL HAUSDORFF SPACES

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ABSTRACT. We show that not every semiregular space is embeddable as an open and dense set of some minimal Hausdorff space. Also a space is constructed for which it is not decidable in Z.F.C. whether such an embedding exists.

1. Introduction. In this paper we investigate the following question.

Question. Is a semiregular space X embeddable in a minimal Hausdorff space as a dense and open subset?

(Note that a Hausdorff space is called *minimal Hausdorff* if it contains no strictly coarser Hausdorff topology and that a space Y is called *semiregular* if $\{\operatorname{int}(\operatorname{cl} A): A \subseteq Y\}$ is an open basis for Y.)

This question appeared in the paper **[Ve]** and was motivated by the following embedding theorem.

THEOREM A. *Let X be a semiregular space. Then*:

(i) **[Ka]** *X is embeddable as a dense subset of a minimal Hausdorff space.*

(ii) **[Ve]** *X is embeddable as an open subset of a minimal Hausdorff space.*

(iii) **[Ve]** *The space $X \oplus X$—two disjoint copies of X—is embeddable as a dense and open subspace of a minimal Hausdorff space.* □

We present two examples which show the following.

EXAMPLE 1. There exists a zero-dimensional Lindelöf space X for which the answer to the question is negative.

EXAMPLE 2. There exists a zero-dimensional Lindelöf space X for which the question cannot be answered in Z.F.C. without additional assumptions.

For these examples we use the following notions.

A function $f\colon X \to Y$ is called *irreducible*, whenever f is surjective and $f(A) \neq Y$, for every closed subset $A \subsetneq X$.

A function $f\colon X \to Y$ is called *θ-continuous* if for each $x \in X$ and each neighborhood U of $f(x)$, there is a neighborhood V of x such that $f(\operatorname{cl} V) \subseteq \operatorname{cl} U$.

The *absolute* of a space X is the unique semiregular and extremally disconnected space EX which can be mapped onto X by a perfect, irreducible and θ-continuous function π.

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If αD is a minimal Hausdorff extension of a discrete space D, then $E\alpha D = \beta D$ and the map π: $E\alpha D \to \alpha D$ is the (unique) θ-continuous extension of id: $D \to \alpha D$ to βD.

The next theorem is the key to the examples.

THEOREM B [**VeW**]. *Let X be a compact space. Let f: $X \to Y$ be a compact and irreducible function onto a set Y. Then the collection $\{f(B)$: B is a closed subset of $X\}$ is a closed base for a topology $\theta(f)$ on Y, the space $(Y, \theta(f))$ is minimal Hausdorff and the function f: $X \to (Y, \theta(f))$ is θ-continuous.*

REMARK. The definitions of all undefined notions we used in the previous theorem can be found in the excellent survey paper [**Wo**] of R. G. Woods.

2. The example. Let D be a discrete space with card $D \geqslant 2^{2^{\omega_0}}$ and let $X = D \cup \{\omega\}$ be the one-point Lindelöffication of D. In particular, the collection $\{D' \cup \{\omega\}$: D' a cocountable subset of $D\}$ is a local base at ω in X.

THEOREM C. *If* card $X > 2^{2^{\omega_0}}$, *then X is not embeddable as a dense and open subset in a minimal Hausdorff space.*

PROOF. Assume the opposite, say X is embedded in the minimal Hausdorff space Y as a dense and open subset.

Then the space Y can be considered as a minimal Hausdorff extension of the discrete space D, say $Y = \alpha D$ and $D \subset X \subset \alpha D$.

Then $E\alpha D = \beta D$ and the absolute function π: $\beta D \to \alpha D$ is the unique θ-continuous extension of id: $D \to \alpha D$ to βD.

We observe the following facts:

(i) Since the function π: $\beta D \to \alpha D$ is perfect and irreducible and the space αD is minimal Hausdorff, the collection $\{\pi(B)$: B is a closed subset of $\beta D\}$ is a closed base for the topology on αD.

(ii) $\pi^{-1}\{\omega\} = \{\mathfrak{F}$: $\mathfrak{F}$ an ultrafilter on D with: $\forall F \in \mathfrak{F}$: card $F > \omega_0\} \subset \beta D - D$.

(iii) If B is a compact subset of βD with $B \cap \pi^{-1}\omega = \varnothing$, then it is easy to see that card $B \leqslant 2^{2^{\omega_0}}$.

In particular, card $\pi^{-1}\{d\} \leqslant 2^{2^{\omega_0}}$ for each $d \in \alpha D - X$.

(iv) Since card $D > 2^{2^{\omega_0}}$, card$(\beta D - D - \pi^{-1}\{\omega\}) > 2^{2^{\omega_0}}$. Since X is an open subset of αD, then by (i), there exists a compact subset $B \subset \beta D$ such that $\alpha D - X \subset \pi(B) \subset \alpha D - \{\omega\}$. However, by (iii), card $B \leqslant 2^{2^{\omega_0}}$ and card $\pi^{-1}\{p\} \leqslant 2^{2^{\omega_0}}$, for each $p \in \alpha D - \{\omega\}$.

Consequently, card $\pi^{-1}(\pi(B)) \leqslant 2^{2^{\omega_0}}$. We conclude, from (iv), that $\beta D - D - \pi^{-1}\{\omega\}$ is not a subset of $\pi^{-1}\pi(B)$, which contradicts that $\alpha D - X \subset \pi(B)$. This completes the proof of the theorem.

THEOREM D. *If* card $X = 2^{2^{\omega_0}}$, *the space X is embeddable as a dense and open subset of a minimal Hausdorff space.*

PROOF. We construct such an extension of X as follows. Consider the space βD and define the closed subset $A \subset \beta D - D$ by

$$A = \{\mathfrak{F}\colon \mathfrak{F} \text{ is an ultrafilter on } D \text{ and card } F > \omega_0 \text{ for each } f \in \mathfrak{F}\}.$$

Note that $\beta D - A = \bigcup\{\text{cl } D' \colon D' \text{ is a countable subset of } D\}$. We conclude that $\text{card}(\beta D - D - A) = (2^{2^{\omega_0}})^{\omega_0} \cdot 2^{2^{\omega_0}} = 2^{2^{\omega_0}}$. Fix a countable set $N \subset D$. Then $\text{cl } N \cap A = \varnothing$ and $\text{card}(\text{cl } N - N) = 2^{2^{\omega_0}}$. Let $g\colon \text{cl } N - N \to \beta D - D - A - \text{cl } N$ be a bijection between these sets. Define a partition E of βD by

$$E = \{\{d\} : d \in D\} \cup \{A\} \cup \{\{x, g(x)\} : x \in \text{cl } N - N\}.$$

Let Y denote the set $\beta D \bmod E$. The corresponding quotient function $f\colon \beta D \to Y$ is a compact and irreducible surjection.

Consider the minimal Hausdorff topology $\theta(f)$ on Y, as defined in Theorem B. The following properties are easy to verify:

(i) $f(D)$ is an open discrete and dense subset of Y,

(ii) the subspace $f(D) \cup \{A\} = f(D \cup A)$ of Y is homeomorphic to the space X,

(iii) $f(D) \cup \{A\}$ is dense in Y (since $f(D) \subset f(D \cup A)$), and

(iv) $f(A) \cup \{A\}$ is open in Y (since $Y - f(D \cup A) = f(\text{cl } N - N)$).

These properties show that we have embedded the space X as a dense and open subset of the minimal Hausdorff space Y.

REMARKS. There are many cardinals χ, e.g. $\chi = 2^{2^{\omega_1}}$, $\chi = \aleph_3$, for which it is consistent to assume that $\chi > 2^{2^{\omega_0}}$ and consistent to assume that $\chi = 2^{2^{\omega_0}}$. Thus, if D is a discrete space of cardinality χ, then X can be embeddable, by Theorem D, or not embeddable, by Theorem C, as a dense and open subspace of some minimal Hausdorff space depending on set-theoretic assumptions of whether $\chi = 2^{2^{\omega_0}}$ or $\chi > 2^{2^{\omega_0}}$.

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