LEBESGUE SETS AND INSERTION OF A CONTINUOUS FUNCTION

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ABSTRACT. Necessary and sufficient conditions in terms of Lebesgue sets are presented for the following two insertion properties for real-valued functions defined on a topological space: (1) If $g < f$ there is a continuous function $h$ such that $g \leq h \leq f$, and for each $x$ for which $g(x) < f(x)$ then $g(x) < h(x) < f(x)$. 
(2) If $g < f$ there is a continuous function $h$ such that $g < h < f$.

1. Statement of results. All functions considered are real-valued. Let $R$ (respectively, $Q$) denote the real (respectively, rational) numbers and write $g \leq f$ (respectively, $g < f$) in case $g(x) \leq f(x)$ (respectively, $g(x) < f(x)$) for each $x$ in the space. For $b$ in $R$ the Lebesgue sets for a function $f$ are defined by $L_b(f) = \{x: f(x) \leq b\}$ and $L^b(f) = \{x: f(x) \geq b\}$. Let $C^*(X)$ denote the lattice of continuous, bounded, and real-valued functions on $X$. The main results of this paper utilize Lebesgue sets to characterize certain insertion properties of a continuous function, and they are based on the following ([2, Theorem 3.5; 8, Theorem 2.1]):

THEOREM 1. If $g < f$ then there is a continuous function $h$ such that $g < h < f$ if and only if for any rational numbers $a$ and $b$ such that $a < b$ the Lebesgue sets $L_b(g)$ and $L^a(f)$ are completely separated.

The following is stated on p. 444 of [11].

THEOREM 2. Let $X$ be a topological space and let $L(X)$ and $U(X)$ be classes of bounded functions defined on $X$ such that any constant function is in $L(X) \cap U(X)$ and such that if $g \in U(X)$, $f \in L(X)$, and $r \in R$ then $g \wedge r \in U(X)$ and $f \vee r \in L(X)$. The following are equivalent:

(i) If $f \in L(X)$, $g \in U(X)$, and $g \leq f$ there exists $h$ in $C^*(X)$ such that $g \leq h \leq f$ and such that $g(x) < h(x) < f(x)$ for each $x$ for which $g(x) < f(x)$.

(ii) If $f \in L(X)$, $g \in U(X)$, and $r \in R$ the Lebesgue sets $L_r(f)$ and $L^r(g)$ are zero sets in $X$.

(iii) If $f \in L(X)$ and $g \in U(X)$ then $f$ (respectively, $g$) is the pointwise limit of and increasing (respectively, decreasing) sequence of continuous functions.

In the situation where $U(X)$ and $L(X)$ are the classes of upper and lower semicontinuous functions, respectively, the equivalence of (i) and (ii) of the above theorem is due to Michael [12], the equivalence of (ii) and (iii) is due to Tong [14], and each of the conditions being equivalent to $X$ is perfectly normal. If $U(X)$ and $L(X)$ are the classes of normal upper and normal lower semicontinuous functions,
respectively, the equivalence of (i), (ii), (iii), and $X$ is an O$_2$ space is established in [8].

Necessary and sufficient conditions in order for a space to satisfy condition (i) of Theorem 2 for general classes of functions are considered in [12, 3, 4, 7, and 10]. Let $B(X)$ denote the Banach lattice of all bounded real-valued functions on a space $X$. If $C$ is a sublattice of the power set of $X$ to which $\emptyset$ and $X$ belong, the smallest convex cone in $B(X)$ that contains the constant functions $1_D, D \in C$, is denoted by $cn(C)$ and its closure by $\overline{cn}(C)$. The results of Blatter and Seever in [3 and 4] require that the classes $U(X)$ and $L(X)$ can be characterized as $\overline{cn}(A)$ and $\overline{cn}(B)$, respectively, for some sublattices $A$ and $B$ of the power set of $X$ and that $A \subset B_0 (= \text{intersection of sequences in } B)$ and $B \subset A_\sigma (= \text{union of sequences in } A)$. The necessary (as proved by Powderly [13]) and sufficient condition of [7] places restrictions on the function $f - g$. These limitations are avoided in Theorem 2.

A portion of the following result is stated on p. 478 of [11].

**THEOREM 3.** Let $L(X)$ and $U(X)$ be classes of bounded real-valued functions on $X$ such that $C^*(X) \subset L(X) \cap U(X)$. The following are equivalent:

(i) For $f \in L(X), g \in U(X)$ and $g < f$ there exists $h \in C^*(X)$ such that $g < h < f$.

(ii) If $f \in L(X), g \in U(X)$ and $g < f$ then for each $r \in Q$ there exist disjoint sets $A_r$ and $B_r$ such that $L_r(f)$ and $X - A_r$ are completely separated, $L_r'(g)$ and $X - B_r$ are completely separated, and each of $\{X - (B_r \cup L_r(f)): r \in Q\}$ and $\{X - (A_r \cup L_r'(g)): r \in Q\}$ covers $X$.

(iii) If $f \in L(X), g \in U(X)$ and $g < f$ then for each rational number $r$ there exist disjoint sets $A_r$ and $B_r$ such that $L_r(f)$ and $X - A_r$ are completely separated, $L_r'(g)$ and $X - B_r$ are completely separated, and $\{X - (A_r \cup B_r): r \in Q\}$ covers $X$.

In the situation in which $L(X)$ and $U(X)$ are the lattices of lower and upper semicontinuous functions, respectively, results of Dowker [5] and Katětov [6] show that a space satisfies (i) of the above theorem if and only if $X$ is normal and countably paracompact. Other specific cases are given in [10, Theorem 4.2]. Necessary and sufficient conditions for a space to satisfy (i) of Theorem 3 for general classes of functions are given in [3, 4, and 7] but these results have restrictions analogous to those mentioned in the discussion following Theorem 2.

It is noted that the bounded condition placed on the functions in Theorems 2 and 3 causes no loss in generality if the properties that define the classes $L(X)$ and $U(X)$ are preserved under an order preserving homeomorphism from $R$ onto a finite interval.

2. Proofs of results. The following lemma is used in combination with Theorem 1 in proving Theorems 2 and 3. The argument is adapted from a technique used in the proof of Theorem 3.3 in [3] and is included here for completeness.

**LEMMA 1.** Let $f, g$ and $k$ be bounded functions such that $g \leq k \leq f$ and $k \in C^*(X)$. If there exist sequences $\{a_n\}$ and $\{b_n\}$ in $C^*(X)$ such that $g \leq a_n$ and $b_n \leq f$ for all $n$, $\inf_n a_n(x) < f(x)$ and $\sup_n b_n(x) > g(x)$ for each $x$ for which $g(x) < f(x)$, then there exists $h$ in $C^*(X)$ such that $g \leq h \leq f$ and for each $x$ for which $g(x) < f(x)$ then $g(x) < h(x) < f(x)$.
PROOF. Using the notation of the lemma, set
\[ h = \sum_{n=1}^{\infty} 2^{-n-1}(a_n \land k + b_n \land k). \]
Since \( g \leq a_n \land k \), \( \sum_{n=1}^{\infty} 2^{-n}g \leq \sum_{n=1}^{\infty} 2^{-n}(a_n \land k) \), or \( g \leq \sum_{n=1}^{\infty} 2^{-n}(a_n \land k) \). Similarly, from \( f \geq a_n \land k \) it follows that \( f \geq \sum_{n=1}^{\infty} 2^{-n}(a_n \land k) \). Thus \( g \leq \sum_{n=1}^{\infty} 2^{-n}(a_n \land k) \leq f \). In the same fashion show that \( g \leq \sum_{n=1}^{\infty} 2^{-n}(k \lor b_n) \leq f \).

From the definition of \( h \) it follows that \( g \leq h \leq f \). Let \( x \) be such that \( g(x) < f(x) \). Choose \( N \) so that \( a_N(x) < f(x) \). Then \( \sum_{n=1}^{\infty} 2^{-n}(a_n \land k)(x) = \sum_{n=N}^{\infty} 2^{-n}(a_n \land k)(x) + 2^{-N}(a_N \land k)(x) < \sum_{n=N}^{\infty} 2^{-n}(a_n \land k)(x) + 2^{-N}f(x) \leq \sum_{n=1}^{\infty} 2^{-n}f(x) = f(x) \).

Similarly, show that \( g(x) < \sum_{n=1}^{\infty} 2^{-n}(b_n \lor k)(x) \). Hence \( g(x) < h(x) < f(x) \) whenever \( g(x) < f(x) \).

That condition (ii) of Theorem 2 implies (i) is a consequence of Proposition 6.1 of [3]. A proof is given here that uses the above lemma since this approach seems considerably more direct.

PROOF OF THEOREM 2. (i)\( \Rightarrow \) (ii) If \( g \in U(X) \) and \( r \in R \) then by hypothesis \( g \land r \in U(X) \) and \( r \in L(X) \). By (i) there is a continuous function \( h \) such that \( g \land r \leq h \leq r \) and if \( g(x) < r \) then \( g(x) < h(x) < r \); \( L^*(g) \) is a zero set since \( L^*(g) = \{ x : h(x) = r \} \). Similarly show that \( L_r(f) \) is a zero set.

(ii)\( \Rightarrow \) (iii) If \( f \) is a lower semicontinuous function defined on a perfectly normal space, Tong’s proof [14] that \( f \) is a pointwise limit of an increasing sequence of continuous functions is based on the Lebesgue set \( L_r(f) \) being a zero set; with trivial modification his proof yields this implication.

(iii)\( \Rightarrow \) (i) Let \( g \in U(X) \) and \( f \in L(X) \) with \( g \leq f \). If \( \{ f_n \} \) is an increasing sequence of continuous functions whose pointwise limit is \( f \) then for any real number \( r \) the Lebesgue set \( L_r(f) \) equals the intersection of the sequence \( \{ L_{r+1/n}(f_n) \} \) of zero sets, and thus \( L_r(f) \) is a zero set. Similarly, use a decreasing sequence \( \{ g_n \} \) of continuous functions whose pointwise limit is \( g \) to show that each \( L^*(g) \) is a zero set. For any rational numbers \( a < b \), \( L^*(g) \) and \( L_a(f) \) are disjoint zero sets and hence are completely separated; by Theorem 1 there is a continuous function \( k \) such that \( g \leq k \leq f \). Since the sequences \( \{ g_n \} \) and \( \{ f_n \} \) satisfy the conditions of Lemma 1, it follows that there exists \( h \) in \( C^*(X) \) such that \( g \leq h \leq f \) and whenever \( g(x) < f(x) \) then \( g(x) < h(x) \leq f(x) \). This concludes the proof of Theorem 2.

If \( k \) maps a space \( X \) into \( R \), call \( k \) regular lower semicontinuous (respectively, regular upper semicontinuous) if for each real number \( r \) the Lebesgue set \( L_r(k) \) (respectively, \( L^*(k) \)) is a regular \( G_\delta \) subset of \( X \). (These functions were considered in [9].) Let \( L(X) \) (respectively, \( U(X) \)) denote the class of bounded regular lower (respectively, upper) semicontinuous functions. The following is an immediate corollary of Theorem 2. If \( g \in U(X) \), \( f \in L(X) \), and \( g \leq f \) there is \( h \) in \( C^*(X) \) such that \( g \leq h \leq f \) and such that \( g(x) < h(x) < f(x) \) whenever \( g(x) < f(x) \) if and only if each regular \( G_\delta \) subset of \( X \) is a zero set. (If \( X \) is an Oz space [1] or if \( X \) is almost normal then each regular \( G_\delta \) subset of \( X \) is a zero set.)

PROOF OF THEOREM 3. (i)\( \Rightarrow \) (iii) If \( f \in L(X) \), \( g \in U(X) \) and \( g < f \) then by (i) there exists \( h \) in \( C^*(X) \) such that \( g < h < f \). Since \( C^*(X) \subset L(X) \cap U(X) \) we may use hypothesis (i) again to show there exist \( h_1 \) and \( h_2 \) in \( C^*(X) \) such that \( g < h_1 < h < h_2 < f \). For each \( r \) in \( Q \) let \( A_r = \{ x : h_2(x) < r \} \) and \( B_r = \{ x : h_1(x) > r \} \). In order to see that \( L_r(f) \) and \( X - A_r \) are completely separated, use (i) to choose \( k \).
in $C^*(X)$ such that $h_2 < k < f$. Since $X - A_r$ and $L_r(k)$ are disjoint zero sets and $L_r(k) \supset L_r(f)$ it follows that $L_r(f)$ and $X - A_r$ are completely separated. Similarly, $L^*(g)$ and $X - B_r$ are completely separated. The sets $X - (A_r \cup B_r)$, $r \in \mathbb{Q}$, cover $X$ since $h_1 < h_2$. Thus (iii) is satisfied.

That (iii) implies (ii) is manifest; the argument to show (ii) implies (i) follows: Let $g \in U(X)$, $f \in L(X)$ and suppose that $-M < g < f < M$. It follows from (ii) that for any rational number $r$ that $L_r(f)$ and $L^*(g)$ are completely separated; in particular for any rationals $a < b$ then $L_a(f)$ and $L^b(g)$ are completely separated. By Theorem 1 there is $k$ in $C^*(X)$ such that $g < k < f$. For each rational number $r$ choose sets $A_r$ and $B_r$ satisfying the conditions of (ii), and then choose $a_r$ and $b_r$ in $C^*(X)$ such that $-M \leq a_r \leq r$, $a_r = -M$ on $L_r(f)$, $a_r = r$ on $X - A_r$, $r \leq b_r \leq M$, $b_r = M$ on $L^*(g)$, and $b_r = r$ on $X - B_r$. If $x \in L_r(f)$ then $a_r(x) = -M < f(x)$ and if $x$ is not in $L_r(f)$ then $f(x) > r \geq a_r(x)$; thus $a_r \leq f$. Let $x \in X$ and choose $s \in \mathbb{Q}$ such that $x$ is $X - (A_s \cup L^s(g))$; since $x \in X - L^s(g)$ then $g(x) < s$ and since $x \in X - A_s$ then $a_s(x) = s$. Thus $\sup r a_r(x) \geq a_s(x) > g(x)$. Similarly, show that $g \leq b_r$ for each $r$ and $\inf r b_r(x) < f(x)$ for each $x$. By Lemma 1 there exists $h$ in $C^*(X)$ such that $g < h < f$. This concludes the proof of Theorem 3.

As one application of Theorem 3 consider the result of Dowker and Katetov mentioned above. Suppose that $X$ is normal and countably paracompact, $g$ is upper semicontinuous, $f$ is lower semicontinuous and $g < f$. Since $\{X - (L_r(f) \cup L^*(g)) : r \in \mathbb{Q}\}$ is a countable open cover of $X$ there is a cover $\{F_r : r \in \mathbb{Q}\}$ of $X$ such that $F_r$ is closed and $F_r \subset X - (L_r(f) \cup L^*(g))$ for each $r$. Since $F_r$ and $L_r(f)$ are completely separated choose $k_r$ in $C^*(X)$ such that $k_r = 0$ on $L_r(f)$, $k_r = 1$ on $F_r$, and let $A_r = \{x : k_r(x) < \frac{1}{2}\}$. Thus $L_r(f)$ and $X - A_r$ are completely separated. Similarly, define $B_r$ so that $L^*(g)$ and $X - B_r$ are completely separated. Since $F_r \subset X - (A_r \cup B_r)$, $\{X - (A_r \cup B_r) : r \in \mathbb{Q}\}$ covers $X$. By Theorem 3 there is $h$ in $C^*(X)$ such that $g < h < f$. This concludes the proof of Theorem 3.

REFERENCES

1. R. L. Blair, Spaces in which special sets are z-embedded, Canad. J. Math. 28 (1976), 673–690.

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