TRIVIALITY OF SIMPLE FIBER-PRESERVING ACTIONS OF TORI ON HILBERT-CUBE-FIBER BUNDLES

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Abstract. Let \( \sigma \) denote the standard based-free action of a torus \( T \) on the Hilbert cube \( Q \). It will be shown that every simple fiber-preserving action of \( T \) on \( Q \times B \), where \( B \) is a finite-dimensional, compact metric absolute retract, is fiber-preserving equivalent to the diagonal action \( \sigma \times \operatorname{id}_B \).

1. Introduction and definitions. Let \( Q \) denote the Hilbert cube \( I_1 \times I_2 \times \cdots \), and let \( G \) be a topological group. An action \( \alpha: G \times M \to M \) of \( G \) on a topological space \( M \) is said to be semifree if, for each \( x \in M \), the isotropy group of \( x \) is either \( G \) or \{e\}. For a finite or compact Lie group \( G \), it is known that the metric cone of \( G \), \( \operatorname{cone}(G) \), is a compact absolute retract (AR). The left translation on \( G \) defines a natural semifree action of \( G \) on \( \operatorname{cone}(G) \) that induces a semifree action, the diagonal action of \( G \) on \( \prod_{n=1}^\infty \operatorname{cone}(G) \approx Q \) [We]. This action has a unique fixed point. Following [B-We], we call it the standard based-free action \( \sigma \) of \( G \) on \( Q \) whose orbit space is denoted by \( Q/\sigma \). An action \( \lambda \) of \( G \) on \( Q \times B \) is said to be fiber-preserving (f.p.) if, for each \( g \in G \), we have \( p_B \circ \lambda(g, \cdot) = p_B(\cdot) \), where \( p_B: Q \times B \to B \) is the projection. Two actions \( \alpha \) and \( \beta: G \times M \to M \) are said to be equivalent if there is a homeomorphism \( f: M \to M \) such that

\[
f(\alpha(g, x)) = \beta(g, f(x))
\]

for all \( x \in M \) and \( g \in G \). A f.p. action \( \lambda \) on \( Q \times B \) is said to be simple if the restriction of \( \lambda \) to each fiber \( Q \times \{b\} \) is equivalent to \( \sigma \) and has fixed point set \( \{(0, b)\} \).

In [L], the following result has been proved: "If \( B \) is a finite-dimensional compact metric AR, then every simple f.p. action of a finite group \( G \) on \( Q \times B \) is equivalent to the diagonal action \( \sigma \times \operatorname{id}_B \)."

The purpose of this paper is to establish a similar result for simple f.p. actions of tori on \( Q \times B \), where \( B \) is a finite-dimensional, compact metric AR, as stated in the abstract or Theorem 6 below.

In this paper, we set \( I = [0, 1] \), \( Q_0 = Q - \{0\} \). For basic notions and results in \( Q \)-manifold theory, we refer to [Ch]. A map \( p: X \to B \) is a Hurewicz fibration if \( p \) has the homotopy covering extension property (HCEP) with respect to every
Let \( p: X \to B \) be a Hurewicz fibration, where \( B \) is a f.d. compact AR and the fibers are \( Q \)-manifolds, then the composition \( p \circ p_X: X \times [0,1) \to X \to B \) is a locally trivial fiber bundle.

Given \( \alpha: G \times Q \times B \to Q \times B \) a simple f.p. action of \( G \) over \( B \), we will define \( E \) to be its orbit space and \( E_0 \) its regular orbit space \( E \setminus \{0\} \times B \) (we identify the fixed point set \( \{0\} \times B \) with its image in the orbit space). Let \( p: E \to B \) denote the natural map induced from the projection map \( p_B: Q \times B \to B \). The fiberwise-reduced cone \( C(E, p) \) of \( E \) over \( B \) is defined to be the quotient \( (E \times \mathbb{I})/\mathcal{D} \), where \( \mathcal{D} \) is the decomposition of \( E \times \mathbb{I} \) whose nondegenerate elements are all closed subsets of the form \( (p^{-1}(b) \times \{1\}) \cup (\partial_b \times \mathbb{I}) \) for \( b \in B \). Let \( d_1 \) denote a metric on \( C(E, p) \) which extends a metric, say \( d_1 \), in \( E \). Note that, besides being endowed with the metric \( d_1 \), the open subset \( E_0 \times [0,1) \) of \( C(E, p) \) also inherits the product metric \( d_2 \) of \( d_1 \) on \( E_0 \) and the usual metric on \( [0,1) \). Given a \( \delta > 0 \), a \( \delta \)-subset of a metric space is a subset of diameter less than \( \delta \).

**Observations.** Let \( K \) be a compact subset of \( E_0 \) and let \( \varepsilon \) be a positive number. Then, there is a \( \delta > 0 \) such that every \( \delta \)-subset of \( (E_0 \times [0,1), d_2) \) intersecting \( K \) is an \( \varepsilon \)-subset of \( (E_0 \times [0,1), d_1) \). Consequently, if \( p: K \times \mathbb{I} \to C(E, p) \) is a \( \delta \)-homotopy (i.e. \( \text{diam}_2 p(x \times \mathbb{I}) < \delta \)) in \( (E_0 \times [0,1), d_2) \) with \( p(x,0) = x \) for all \( x \) in \( K \), then \( \text{diam}_1 p(x \times \mathbb{I}) < \varepsilon \) in \( (E_0 \times [0,1), d_1) \) for all \( x \) in \( K \). In particular, for the projection \( p_{E}: C(E, p) \to E \) and a \( \delta \)-subset \( A \) of \( (E_0 \times [0,1), d_2) \) intersecting \( K \), then \( \text{diam}_1 p_{E}(A) < \varepsilon \) in \( (E, d_1) \). Similar properties also hold true when \( d_1 \) and \( d_2 \) are interchanged.

Given an open cover \( \beta \) of a space \( Y \), a homomotopy \( H: X \times \mathbb{I} \to Y \) is said to be \( \beta \)-homotopy if for each \( x \in X \), the track \( H(x \times \mathbb{I}) \) is contained in a member of \( \beta \); a map \( f: X \to Y \) is said to be a \( \beta \)-equivalence if there is a map \( g: Y \to X \) such that \( f \circ g \) is \( \beta \)-homotopic to \( \text{id}_Y \) and \( g \circ f \) is \( f^{-1}(\beta) \)-homotopic to \( \text{id}_X \), where \( f^{-1}(\beta) \) denotes the open cover \( \{f^{-1}(U) \mid U \in \beta \} \) of \( X \).

**2. Results and details of proofs.** In the proof of the following proposition, we refer to p. 266 of [Do] for the definition of section extension property (SEP), numerable open covering, etc.

**Proposition 1.** If \( \lambda \) is a simple f.p. action of a finite group or a compact Lie group \( G \) on \( Q \times B \), then the natural map \( p: E_0 \to B \) is a Hurewicz fibration.

**Proof.** Since \( E_0 \) and \( B \) are metric spaces, we only have to show that \( p \) has the HCEP for the class of all metric spaces [Du, Corollary 2.3, p. 396]. Assume that \( X \) is a metric space. Given a commutative diagram

\[
\begin{array}{ccc}
X \times \{0\} & \xrightarrow{h} & E_0 \\
\downarrow & & \downarrow p \\
X \times \mathbb{I} & \xrightarrow{\phi} & B
\end{array}
\]
we will show that \( \phi \) has a lifting \( \Phi: X \times I \to E_0 \) with \( \Phi(x, 0) = h(x) \) for all \( x \) in \( X \). Equivalently, if we define

(1) \( R = \{ (x, w) \in X \times E_0' \mid w(0) = h(x) \text{ and } pw(t) = \phi(x, t) \} \), and

(2) \( q: R \to X \) by \( q(x, w) = x \),

then we will show that there is a numerable open cover \( \{ V_\lambda \mid \lambda \in \Lambda \} \) of \( X \) such that the restriction \( q_\lambda = q \mid q^{-1}(V_\lambda) : q^{-1}(V_\lambda) \to V_\lambda \) has SEP; hence, \( q \) has a cross-section by Theorem 2.7 of [Do].

Let \( \{ U_\lambda \mid \lambda \in \Lambda \} \) be a covering of \( E_0 \) consisting of contractible open subsets. Let \( V_\lambda = h^{-1}(U_\lambda), \lambda \in \Lambda \), then \( \{ V_\lambda \mid \lambda \in \Lambda \} \) is numerable since \( X \) is a metric space. Let \( \pi: Q_0 \times B \to E_0 \) denote the orbit map. Then, \( \pi \mid \pi^{-1}(U_\lambda) : \pi^{-1}(U_\lambda) \to U_\lambda \) is a trivial fiber bundle with fiber \( G \) [Br, Theorem 5.8, p. 88]; hence, there is a lifting \( \tilde{h}_\lambda: V_\lambda \times \{ 0 \} \to \pi^{-1}(U_\lambda) \subset Q_0 \times B \) of \( h_\lambda = h \mid (V_\lambda \times \{ 0 \}) \). Observe that the pair \((\phi_\lambda, \tilde{h}_\lambda)\), where \( \phi_\lambda = \phi \mid (V_\lambda \times I) \), has the HCEP; consequently, so does the pair \((\phi_\lambda, h_\lambda)\). Therefore, the map \( q_\lambda: q^{-1}(V_\lambda) \to V_\lambda \) has SEP by Lemma 4.5 of [Do]. So, the proof of the proposition is complete.

For the sake of simplicity, let \( F_h \) denote \( p^{-1}(b) \), \( X_h \) its reduced cone at \((0, b)\) and \( F_{h,b} \) the regular orbit space \( F_h \cap E_0 \) over \( b \) for each \( b \in B \); let \( Y_h \) denote \((Q_0/\sigma) \times \{ b \}\). Let \( d \) denote the product metric on \( Q_0/\sigma \times B \). Given \( b, c \in B \), let \( \epsilon_{h,c}: Y_h \to Y_b \) be the natural homeomorphism defined by \( \epsilon_{h,c}(x, b) = (x, c) \). Then \( d(\epsilon_{h,c}(x, b), (x, c)) = d_p(b, c) \) for each \((x, b) \in Y_h\).

From the above proposition and Lemma 0, it follows that the composition \( p \circ p_{E_0}: E_0 \times [0,1) \to B \) is a trivial fiber bundle whose fiber is homeomorphic to \((Q_0/\sigma) \times B \) [B-We]. Therefore, \( E_0 \times [0,1) \) is f.p. homeomorphic to \((Q_0/\sigma) \times B \) over \( B \). Consequently, its fiberwise-one-point compactification \( C(E, p) \) is f.p. homeomorphic to \((Q_0/\sigma) \times B \) over \( B \). Let \( h \) denote such a homeomorphism in the following Lemmas 2–5.

**Lemma 2.** Given an \( \epsilon > 0 \), there is a \( \delta > 0 \) such that if \( d_p(b, c) < \delta \), then \( d_i(h^{-1}\epsilon_{h,c}(h(x), x)) < \epsilon \) for all \( x \in X_h \) and \( d_i(h^{-1}\epsilon_{h,c}(h(z), z)) < \epsilon \) for all \( z \in X_c \).

**Proof.** From the uniform continuity of \( h^{-1} \), there is a \( \delta > 0 \) \((\delta < \epsilon)\) such that \( d(h^{-1}(y), h^{-1}(y')) < \epsilon \) for every pair of \( y, y' \in (Q_0/\sigma) \times B \) with \( d(y, y') < \delta \). Now, if \( d_p(b, c) < \delta \), then for each \( x \in X_h \) we have \( d_i(h_{h,c}(h(x), h(x)) = d_p(b, c) < \delta \); hence, \( d_i(h^{-1}\epsilon_{h,c}(h(x), x)) = d_i(h^{-1}\epsilon_{h,c}(h(x), h^{-1}(h(x))) < \epsilon \) as we desired. The proof of the second inequality is the same.

**Lemma 3.** Given an \( \epsilon > 0 \), there is a \( \delta \) \((0 < \delta < \epsilon)\) such that if \( d_p(b, c) < \delta \) and if \( A \) is a \( \epsilon \)-subset of \((X_h, d_i)\), then \( h^{-1}\epsilon_{h,c}(h(A)) \) is an \( \epsilon \)-subset of \((C(E, p), d_i)\).

**Proof.** From Lemma 1, choose a \( \delta < \epsilon/3 \) such that if \( d_p(b, c) < \delta \), then \( d_i(h^{-1}\epsilon_{h,c}(h(x), x)) < \epsilon/3 \) for each \( x \in X_h \). Then, for \( x, y \in X_h \) with \( d_i(x, y) < \delta \), we have
\[
d_i(h^{-1}\epsilon_{h,c}(h(y)), h^{-1}\epsilon_{h,c}(h(x))) \leq d_i(h^{-1}\epsilon_{h,c}(h(x), x) + d_i(x, y) + d_i(y, h^{-1}\epsilon_{h,c}(h(y))) < \epsilon.
\]
**Lemma 4.** Fix a point \( b \in B \) and a compact subset \( K \) of \( F_h - \{b\} \). Given an \( \varepsilon > 0 \), then there is a \( \delta > 0 \) such that if \( d_B(b, c) < \delta \), there are maps \( f : F_c \to F_h \) and \( g : F_h \to F_c \) having the following properties:

(a) \( d_1(f(x), x) < \varepsilon \) for all \( x \in f^{-1}(K) \), and \( d_1(g(y), y) < \varepsilon \) for all \( y \in K \).

(b) \( g \circ f \mid f^{-1}(K) \) is \( \varepsilon \)-homotopic to the inclusion \( f^{-1}(K) \subset F_c \).

(c) \( f \circ g \mid K \) is \( \varepsilon \)-homotopic to the inclusion \( K \subset F_h \).

**Sublemma.** Let \( M \) be a compact subset of \( E_0 \). Given an \( \varepsilon > 0 \), there is an \( \delta_1 < \varepsilon \) such that if \( d_B(b, c) < \delta_1 \) and if \( A \) is an \( \varepsilon_1 \)-subset of \( (X_c, d_1) \) intersecting \( M \), then \( p_{E_0} h^{-1} \varepsilon_{c,b} h(A) \) is an \( \varepsilon \)-subset of \( F_c \).

**Proof.** Let \( \eta \) be a positive number such that the closure \( P \) of the \( \eta \)-neighborhood \( N_\eta(M) \) of \( M \) is a compact subset of \( E_0 \). Choose \( \mu_1 \) (\( 0 < \mu_1 < \eta \)) such that \( \text{diam}_1 p_{E_0}(R) < \varepsilon \) for each \( \mu \)-subset \( R \) of \( (P \times [0, \mu], d_1) \). Then, choose \( \varepsilon_1 (0 < \varepsilon_1 < \mu_1/2) \) such that if \( d_R(b, c) < \varepsilon_1 \), then

\[
(*) \quad \text{diam}(h^{-1} \varepsilon_{c,b} h(A)) < \mu_1/2 \quad \text{with respect to } d_1 \quad \text{and } d_2 \quad \text{for every } \varepsilon_1\text{-subset } A \quad \text{of} \quad (X_c, d_1) \quad \text{intersecting } M \quad \text{(see Lemma 3 and Observations)}, \quad \text{and} \quad

\[
(**) \quad d_1(h^{-1} \varepsilon_{c,b} h(x), x) < \mu_1/2 \quad \text{and} \quad d_2(h^{-1} \varepsilon_{c,b} h(x), x) < \mu_1/2 \quad \text{for every } x \in X_c \cap M \quad \text{(see Lemma 2 and Observations)}.

Now, if \( A \) is an \( \varepsilon_1 \)-subset of \( (X_c, d_1) \) intersecting \( M \), then \( h^{-1} \varepsilon_{c,b} h(A) \) intersects \( N_{\mu_1/2}(M) \times [0, \mu_1/2] \) by (**). Combining this fact and (*), we observe that the set \( R = h^{-1} \varepsilon_{c,b} h(A) \) is a \( \mu_1 \)-subset of \( (N_{\mu_1}(M) \times [0, \mu], d_1) \subset (P \times [0, \mu], d_1) \). Therefore, \( \text{diam}_1(p_{E_0} h^{-1} \varepsilon_{c,b} h(A)) < \varepsilon \) by the choice of \( \mu \).

Let us return to the proof of Lemma 4. Given an \( \varepsilon > 0 \), we will choose inductively the positive numbers \( \varepsilon > \varepsilon_1 > \delta_2 > \delta_1 > \delta_E > \delta \) as follows: We can assume that \( \varepsilon \) is so small that \( N_{\varepsilon}(K) \cap N_{\varepsilon}(B) = \emptyset \), and let \( M \) be the closure of \( N_{\varepsilon}(K) \).

(1) Choose \( \varepsilon_1 \), from the Sublemma, such that if \( H : Z \times I \to (E_0 \times [0, 1], d_1) \) is an \( \varepsilon_1 \)-homotopy with \( H(z \times I) \cap M \neq \emptyset \) and \( H(z \times I) \subset X_h \) for all \( z \in Z \), then \( (p_{E_0} h^{-1} \varepsilon_{c,b} h) \circ H \) is an \( \varepsilon \)-homotopy in \( (E, d_1) \) when \( d_B(b, c) < \varepsilon_1 \).

(2) Choose \( \delta_2 \) such that every \( \delta_2 \)-homotopy \( F : Z \times I \to (E_0 \times [0, 1], d_2) \) with \( F(z \times I) \cap M \neq \emptyset \) for all \( z \in Z \) is an \( \varepsilon_1 \)-homotopy in \( (E_0 \times [0, 1], d_1) \) (from Observations).

(3) Choose \( \delta_1 \) such that every \( \delta_1 \)-subset of \( (E_0 \times [0, 1], d_1) \) intersecting \( M \) is a \( \delta_2 \)-subset of \( (E_0 \times [0, 1], d_2) \) (from Observations).

(4) Choose \( \delta_E \) such that if \( A \) is a \( \delta_E \)-subset of \( (E_0 \times [0, 1], d_1) \) intersecting \( M \), then \( p_{E_0}(A) \) is a \( \delta_E \)-subset of \( (E, d_1) \).

(5) Choose \( \delta \) such that if \( d_B(b, c) < \delta \), then \( d_1(h^{-1} \varepsilon_{c,b} h(x), x) < \delta_E \) for all \( x \in X_c \) and \( d_1(h^{-1} \varepsilon_{c,b} h(y), y) < \delta_E \) for all \( y \in X_h \) (from Lemma 2).

With this choice of \( \delta \), we now define \( f = p_{E_0} h^{-1} \varepsilon_{c,b} h \) and \( g = p_{E_0} h^{-1} \varepsilon_{b,c} h \) and show that \( f \) and \( g \) have all required properties if \( d_B(b, c) < \delta \).

(a) For \( x \in f^{-1}(K) \subset F_c \subset X_c \), then \( d_1(h^{-1} \varepsilon_{c,b} h(x), x) < \delta_E \) by (5); hence, \( d_1(p_{E_0} h^{-1} \varepsilon_{c,b} h(x), p_E(x)) < \delta_1/2 \) by (4); consequently, \( d_1(f(x), x) < \delta_1/2 < \varepsilon \).

Similarly, \( d_1(g(y), y) < \varepsilon \) for \( y \in K \subset F_h \).
(b) Observe that if \( x \in f^{-1}(K) \), then
\[
d_1(p_E^{-1}e_{c,b}h(x), h^{-1}e_{c,b}h(x)) = d_1(p_E^{-1}e_{c,b}h(x), x) + d_1(x, h^{-1}e_{c,b}h(x))
\]
\[
= d_1(f(x), x) + d_1(x, h^{-1}e_{c,b}h(x))
\]
\[
< \delta_1/2 + \delta_E < \delta_1
\]
(from the proof of (a) and by (5)). Hence, \( d_2(f(x), h^{-1}e_{c,b}h(x)) < \delta_2 \) by (3), since \( f(x) \in M \). Therefore, \( f \) is \( \delta_2 \)-homotopic along \([0,1)\) to \( h^{-1}e_{c,b}h \) in \((F_{b,0} \times [0,1), d_2) \subset X_b \), say \( H \). Consequently, \( H \) is an \( \epsilon \)-homotopy in \((X_b, d_1)\) by (2). Finally, \( (p_E^{-1}e_{b,c}h) \circ H \) is an \( \epsilon \)-homotopy by (1) from \( g \circ f \) to
\[
p_E^{-1}e_{b,c}h_{h^{-1}e_{c,b}h} = p_E|f^{-1}(K) \subset F_c.
\]
(c) The proof of (c) is the same as that of (b).

The proof of Lemma 4 is now complete.

**Lemma 5.** Fix an element \( b \) of \( B \). Given an open cover \( \beta \) of \( F_b \) and an open subset \( U \) of \( F_b \) whose closure \( K \) is contained in \( F_{b,0} \), then there is a \( \delta > 0 \) such that, if \( c \in B \) with \( d_B(b, c) < \delta \), then there is a map \( f: F_c \to F_b \) which is a \( \beta \)-equivalence over \( U \).

**Proof.** Let \( \lambda \) be a positive number such that every \( \lambda \)-subset of \( F_b \) intersecting the compactum \( K \) is contained in a member of \( \beta \) meeting \( K \). Let \( \delta \) be a positive number from Lemma 4 corresponding to \( \epsilon = \lambda/4 \); and let \( f \) and \( g \) be two maps satisfying (a), (b) and (c) in Lemma 4.

Now, it is clear that \( f \circ g | K \) is \( \beta \)-homotopic to the inclusion \( K \subset F_{b,0} \) by (c). Moreover, there is a \( (\lambda/4) \)-homotopy \( H \) from \( g \circ f | f^{-1}(K) \) to the inclusion \( f^{-1}(K) \subset F_{c,0} \) by (b); then \( f \circ H \) is a \( (3\lambda/4) \)-homotopy by (a), and it is a \( \beta \)-homotopy since each tract intersects \( K \). Therefore, \( f \) is a \( \beta \)-equivalence over \( U \) as we desired.

**Theorem 6.** Let \( B \) be a finite-dimensional compact metric AR. If \( \lambda \) is a simple f.p. action of the torus \( T^n \) on \( Q \times B \), then (1) the map \( p: E \to B \) is a locally trivial fiber bundle, and (2) \( \lambda \) is f.p. equivalent to the diagonal action \( \sigma \times \text{id}_K \).

**Proof.** The proof will be similar to that of Theorem 3.2 in [L].

(1) Since the homeomorphism group \( \text{Homeo}(Q/\sigma) \) is locally contractible [L, Theorem 2.2], we will only have to show that the map \( p \) is completely regular; i.e. given an element \( b \in B \) and an \( \epsilon > 0 \), we will find a \( \delta > 0 \) such that, for each \( c \in B \) with \( d_B(b, c) < \delta \), there is a homeomorphism \( \phi: F_c \to F_b \) such that \( d_1(\phi(x), x) < \epsilon \) for each \( x \in F_c \).

Following [B-We], we will use the following notation in this proof. Let \( L_n \) denote the orbit space \((T \times T \times \cdots \times T)/\alpha\) of the induced diagonal action \( \alpha \) (from the left translation of \( T \) on itself) on the join of \( n \) copies of \( T \). Then, \( L_n \) is naturally embedded in \( L_{n+1} \), and let \( M_n \) denote the mapping cylinder
\[
\text{Map}(L_1 \to L_2 \to \cdots \to L_n).
\]

Define \( M_\infty = \bigcup_1^\infty M_n \). It is proved in [B-We] that \( F_b \) is homeomorphic to the one-point compactification of \( M_\infty \times Q \).
Given an $\epsilon > 0$, there is an integer $m$ such that $F_b - (M_m \times Q)$ is contained in $N_{\epsilon/2}(0, b)$, the $(\epsilon/2)$-neighborhood of $(0, b)$. Let $\epsilon' (0 < \epsilon' < \epsilon/2)$ be chosen such that $M_{m+2} \times Q$ misses $N_{\epsilon'}(0, b)$.

Let $\beta$ be an open cover of $F_b$ as in Theorem 3.6 in [F] such that every $\beta$-equivalence $f: N \rightarrow F_{b,0}$ ($N$ is a $Q$-manifold) over the interior of $M_{m+2} \times Q$ is $(\epsilon'/2)$-homotopic to a map $\tilde{f}: N \rightarrow F_{b,0}$ having the following properties:

(a) $\tilde{f} | f^{-1}(M_{m+1} \times Q)$ is an open embedding, and
(b) $f(x) = \tilde{f}(x)$ for all $x \in N - f^{-1}(M_{m+2} \times Q)$.

Now, choose $\delta (0 < \delta < \epsilon'/2)$ as in Lemma 5 corresponding to the open cover $\beta$ and $U = \text{int} M_{m+2} \times Q$. Let $b$ and $c$ be in $B$ with $d_{m}(b, c) < \delta$, and let $f: F_c \rightarrow F_b$ be a $\beta$-equivalence given by Lemma 5, and let $\tilde{f}$ be as in the above paragraph. Observe that $\tilde{f}^{-1}(L_{m+1/2} \times Q)$ is bicollared in $F_{c,0}$ and it is contained in $N(0, c)$. Then, as in the proof of Theorem 3.2 in [L], since $F_{c,0}$ and $F_{b,0}$ are of the homotopy type of the Eilenberg-Mac Lane space $K(\mathbb{Z}, 2)$, we can extend $\tilde{f} | f^{-1}(M_{m+1/2} \times Q)$ to a homeomorphism $\phi: F_{c,0} \rightarrow F_{b,0}$, and finally from $F_c$ onto $F_b$. Finally, it can be verified that $d(\phi(x), x) < \epsilon$ for all $x \in F_c$. Therefore, the proof of (1) is complete.

(2) By (1), we now identify $E_0$ with $(Q_0/\sigma) \times B$. Since the principal $T$-bundle $p_1: Q_0 \times B \rightarrow E_0 = (Q_0/\sigma) \times B$ is universal and $B$ is contractible the action is classified by the projection map $E_0 = Q_0/\sigma \times B \rightarrow Q_0/\sigma \times \{b_0\}$ for any $b_0 \in B$ and is therefore equivalent to $\sigma \times \text{id}_B$ by a fiber-preserving, equivariant homeomorphism $g: Q_0 \times B \rightarrow Q_0 \times B$, which must extend to a homeomorphism of $Q \times B$.

REFERENCES


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