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SHORTER NOTES

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SCHUR'S GENERALIZATION OF HILBERT'S INEQUALITY

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We assume a_m, b_n, c_k are complex constants, m, n, λ_k are integers, and α is real but not an integer. If Σ denotes a finite sum, clearly

$$\int_0^{2\pi} \left| \sum c_k e^{i\lambda_k x} \right| dx \geq \left| \int_0^{2\pi} \sum c_k e^{i(\lambda_k - \alpha)x}\, dx \right| = 2\,|\sin \pi\alpha| \left| \sum \frac{c_k}{\lambda_k - \alpha} \right|.$$

On the other hand, by the Schwarz inequality and the orthogonality of $\{e^{inx}\}$,

$$\int_0^{2\pi} \left| \sum a_m e^{imx} \sum b_n e^{inx} \right| dx \leq \left(2\pi \sum |a_m|^2\right)^{1/2} \left(2\pi \sum |b_n|^2\right)^{1/2}.$$

Since the left member of this second inequality admits a representation as the left member of the first, with $c_k = a_m b_n$, $\lambda_k = m + n$, and k running over the indices (m, n), the two together give Schur's form of Hilbert's inequality,

$$\left| \sum \frac{a_m b_n}{m + n - \alpha} \right| \leq \frac{\pi}{|\sin \pi\alpha|} \left(\sum |a_m|^2\right)^{1/2} \left(\sum |b_n|^2\right)^{1/2}.$$

The method succeeds with equal ease, and gives the same result, when a_m, b_n, c_k are square matrices of the same size over $\mathbf{C}$ and $|\cdot|$ denotes the Euclidean norm; as far as we know, this result is new. Further extensions involve additional ideas, however, and will be presented elsewhere.

The above method has affinities with those in [**1–3**] but does not assume $m + n - \alpha > 0$ and does not require use of a polarization identity to pass from the case $a_m = b_m$ to the general case.

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