SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is no other outlet.

SCHUR'S GENERALIZATION OF HILBERT'S INEQUALITY

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We assume $a_m, b_n, c_k$ are complex constants, $m, n, \lambda_k$ are integers, and $\alpha$ is real but not an integer. If $\Sigma$ denotes a finite sum, clearly

$$
\int_0^{2\pi} \left| \sum c_k e^{i\lambda_k x} \right| dx \geq \left| \sum c_k e^{i(\lambda_k - \alpha)x} \right| = 2 \left| \sin \alpha \right| \left| \sum \frac{c_k}{\lambda_k - \alpha} \right|.
$$

On the other hand, by the Schwarz inequality and the orthogonality of $\{e^{inx}\}$,

$$
\int_0^{2\pi} \left| \sum a_m e^{inx} \right| \left| \sum b_n e^{inx} \right| dx \leq (2\pi \sum |a_m|^2)^{1/2} (2\pi \sum |b_n|^2)^{1/2}.
$$

Since the left member of this second inequality admits a representation as the left member of the first, with $c_k = a_m b_n$, $\lambda_k = m + n$, and $k$ running over the indices $(m, n)$, the two together give Schur's form of Hilbert's inequality,

$$
\left| \sum \frac{a_m b_n}{m + n - \alpha} \right| \leq \frac{\pi}{\sin \alpha} \left( \sum |a_m|^2 \right)^{1/2} \left( \sum |b_n|^2 \right)^{1/2}.
$$

The method succeeds with equal ease, and gives the same result, when $a_m, b_n, c_k$ are square matrices of the same size over $\mathbb{C}$ and $| \cdot |$ denotes the Euclidean norm; as far as we know, this result is new. Further extensions involve additional ideas, however, and will be presented elsewhere.

The above method has affinities with those in [1–3] but does not assume $m + n - \alpha > 0$ and does not require use of a polarization identity to pass from the case $a_m = b_m$ to the general case.

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