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A CHARACTERIZATION OF WARFIELD GROUPS[1]

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ABSTRACT. Local Warfield groups are generalizations of totally projective p-groups. This paper presents a characterization of local Warfield groups which is the analogue of the description of totally projective p-groups as groups with a nice composition series.

All groups considered in this paper will be p-local Abelian groups; that is, modules over Z_p, the integers localized at p, where p is a prime. We will use valuated groups throughout, adopting the convention that a subgroup of a valuated group will always carry the induced valuation. The category of p-local valuated Abelian groups was studied by Richman and Walker in **[5]** and we will use their notation and definitions.

Warfield groups were introduced by Warfield in **[7]**, where he defined numerical invariants, later named Warfield invariants, for this class of groups. He used these invariants to prove a generalized version of Ulm's theorem; namely, two Warfield groups are isomorphic if and only if they have the same Ulm and Warfield invariants. Stanton **[6]** later extended these invariants to all groups. An alternate development of Warfield's theory was given by Hunter, Richman and Walker in **[3]**. Although an error in Warfield's proof of the isomorphism theorem was recently found, the theory remains intact since this was avoided in the alternate development. Hunter, Richman and Walker also have an existence theorem for Warfield groups **[2]**. These results have made Warfield groups very important in the study of p-local abelian groups.

One of the outstanding features of the theory of totally projective p-groups is the relatively large collection of characterizations which provide strikingly different approaches to the subject. Since Warfield groups are generalizations of totally projective p-groups, it is only natural to look for characterizations of Warfield groups which parallel those for totally projective p-groups. Some of these are already known. For example, totally projective p-groups can be characterized as simply presented p-groups, while Warfield groups are summands of simply presented groups **[7]**. Also, totally projective p-groups are the projectives relative to balanced exact sequences, and Warfield groups are the projectives relative to the sequentially

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pure exact sequences [7]. The purpose of this paper is to present another characterization of Warfield groups; one which is the analogue of the description of totally projective p-groups as groups with a nice composition series.

We begin by giving some necessary definitions. If A is a subgroup of a valuated group B, we say that A is *nice* in B if every coset of A has an element of maximal value. In other words, for every $b \in B$ there exists an $a \in A$ so that $v(b+a) = v(b+A)$. This is a natural extension of the concept of a nice subgroup in Abelian groups, but we point out that in the valuated setting, the short exact sequences $A \subset B \to C$ where A is nice in B, form a proper class [**5**]. We refer the reader to [**4**] for a definition of proper class. In particular, it is important to note that this means that niceness is a transitive property.

A *decomposition basis* X for a valuated group G is a basis for a direct sum of valuated cyclics in G so that $G/\langle X \rangle$ is torsion. If $\langle X \rangle$ is nice in G, then we say that X is a *nice decomposition basis* of G. A *Warfield group* is a group G with a nice decomposition basis X so that $G/\langle X \rangle$ is totally projective.

A *value sequence* $\mu = (\mu_0, \mu_1, \dots)$ is a sequence of ordinals and symbols ∞ satisfying $\mu_i < \mu_{i+1}$ for all i, with the convention that $\infty < \infty$. Multiplication by p is defined by $p\mu = (\mu_1, \mu_2, \dots)$. If ν is another value sequence, we say that $\nu \geqslant \mu$ if $\nu_i \geqslant \mu_i$ for every i. If x is an element of a valuated group, the *value sequence of* x is $V(x) = (v(x), v(px), \dots)$.

Let A be a nice subgroup of a valuated group B and let f denote the quotient map $B \to B/A$. As in [**3**], we say that a is *quasi-sequentially nice* in B if for each $b \in B$, there exists an integer n and an element $b' \in B$ so that $f(b') = f(p^n b)$ and $V(b') = V(f(b'))$. Notice that if A is a nice subgroup of B and B/A is torsion, then A is quasi-sequentially nice in B. Many of the useful properties of quasi-sequentially nice subgroups are contained in the following proposition whose proof is routine. We will need these properties in the proof of the theorem.

PROPOSITION. *The collection of short exact sequences $A \subset B \to C$ where A is quasi-sequentially nice in B is a proper class.*

With a few more definitions, we will be ready for the characteristic of Warfield groups. A *smooth chain* of groups is a well-ordered, strictly ascending chain of groups

$$0 = G_0 \subset G_1 \subset \cdots \subset G_\alpha \subset G_{\alpha+1} \subset \cdots \subset G_\gamma$$

where $G_\beta = \bigcup_{\alpha<\beta} G_\alpha$ whenever β is a limit ordinal. A *composition series* (for groups) is a smooth chain of groups G_α, where $G_{\alpha+1}/G_\alpha$ is cyclic of order p for each α. For the result which follows, we relax the condition on quotients to allow for infinite cyclics. Thus we will say that a smooth chain of groups G_α *has cyclic quotients* if $G_{\alpha+1}/G_\alpha$ is cyclic for all α.

THEOREM. *A group is a Warfield group if and only if it is the union of a smooth chain of quasi-sequentially nice subgroups with cyclic quotients.*

PROOF. Let G be a Warfield group. Then G has a nice decomposition basis X so that $G/\langle X \rangle$ is totally projective. For some ordinal λ, we may write $X = \{x_\alpha \mid \alpha < \lambda\}$.

For each α, define $M_\alpha = \bigoplus_{\gamma<\alpha}\langle x_\gamma \rangle$. These subgroups of $\langle X \rangle$ are quasi-sequentially nice in $\langle X \rangle$ since they are summands. Thus

$$0 = M_0 \subset M_1 \subset \cdots \subset M_\lambda = \langle X \rangle$$

is a smooth chain of quasi-sequentially nice subgroups of $\langle X \rangle$ with cyclic quotients.

Since $G/\langle X \rangle$ is totally projective, it has a nice composition series which we will denote by

$$0 = N_0/\langle X \rangle \subset N_1/\langle X \rangle \subset \cdots \subset N_\beta/\langle X \rangle = G/\langle X \rangle.$$

For each α, we have $N_{\alpha+1}/N_\alpha$ cyclic. Also, $\langle X \rangle$ is quasi-sequentially nice in G since $G/\langle X \rangle$ is torsion. Thus N_α is quasi-sequentially nice in G. Now by setting $M_{\lambda+\alpha} = N_\alpha$, we see that G is the union of a smooth chain

$$0 = M_0 \subset \cdots \subset M_\lambda \subset M_{\lambda+1} \subset \cdots \subset M_{\lambda+\beta} = G$$

of quasi-sequentially nice subgroups with cyclic quotients.

To prove sufficiency, let G be a group which is the union of a smooth chain of quasi-sequentially nice subgroups with cyclic quotients, which we will denote by

$$0 = N_0 \subset N_1 \subset \cdots \subset N_\lambda = G.$$

For each α, let f_α denote the quotient map $N_{\alpha+1}/N_\alpha$. Let $f_\alpha(z_\alpha)$ be a generator of $N_{\alpha+1}/N_\alpha$. Since N_α is quasi-sequentially nice in $N_{\alpha+1}$, there is an integer m and an element $x_\alpha \in N_{\alpha+1}$ so that $f_\alpha(p^m z_\alpha) = f_\alpha(x_\alpha)$ and $V(x_\alpha) = V(f_\alpha(x_\alpha))$. If $N_{\alpha+1}/N_\alpha$ is torsion, choose $x_\alpha = 0$. Define $X = \{x_\alpha \mid x_\alpha \neq 0\}$. We will prove that X is a nice decomposition basis of G and $G/\langle X \rangle$ is totally projective.

We begin with an observation which is useful in the arguments to follow. If $a \in N_\alpha$ and $y = \sum_{\beta \geqslant \alpha} t_\beta x_\beta$, then $v(a+y) \leqslant v(a)$. The proof of this proceeds by induction on the largest index γ used in this sum. By the choice of x_γ, we know that $v(a+y) \leqslant v(t_\gamma x_\gamma)$. Thus $v(a+y-t_\gamma x_\gamma) \geqslant v(a+y)$. However, by induction, $v(a+y-t_\gamma x_\gamma) \leqslant v(a)$ so that $v(a+y) \leqslant v(a)$.

Next we show that $\langle X \rangle = \bigoplus \langle x_\alpha \rangle$. Let x be an arbitrary element of $\langle X \rangle$ and write $x = \sum t_\alpha x_\alpha$. It suffices to prove that $v(x) \leqslant \min\{v(t_\alpha x_\alpha)\}$. Let γ be the smallest index used in this representation. Then, by the observation above, $v(x) \leqslant v(t_\gamma x_\gamma)$. Hence $v(x - t_\gamma x_\gamma) \geqslant v(x)$ and by induction on the number of nonzero components of x, we know $v(x - z_\gamma x_\gamma) = \min\{v(t_\alpha x_\alpha) \mid \gamma < \alpha\}$. Thus $v(x) \leqslant \min\{v(t_\alpha x_\alpha)\}$.

We must also prove that $G/\langle X \rangle$ is torsion. Let g be an element of G and let $\alpha + 1$ be the smallest ordinal such that $g \in N_{\alpha+1}$. There is an r so that $f_\alpha(g) = f_\alpha(rz_\alpha)$. Therefore, for some m we have $f_\alpha(p^m g) = f_\alpha(rx_\alpha)$. Since $p^m g - rx_\alpha \in N_\alpha$, by induction on α, there is an s so that $s(p^m g - rx_\alpha) \in \langle X \rangle$. So $sp^m g \in \langle X \rangle$.

To show that $\langle X \rangle$ is nice in G, we will prove the stronger statement that $\langle X \rangle + N_\alpha$ is nice in $\langle X \rangle + N_{\alpha+1}$, which will also be useful in the final step of the proof. Then by induction and the transitivity of niceness, $\langle X \rangle$ is nice in G. Let $a \in N_{\alpha+1}$. We may dispense with the simple case where $v(a + N_\alpha) \geqslant v(x_\alpha + N_\alpha) \neq \infty$. In this case, there must be an r so that $a - rx_\alpha \in N_\alpha$. Then

$$v(a - (a - rx_\alpha) - rx_\alpha) = \infty = v(a + \langle X \rangle + N_\alpha).$$

If $v(a+N_\alpha) < v(x_\alpha + N_\alpha)$, we need to find a maximum among $\{v(a+x+n) \mid x \in \langle X \rangle$ and $n \in N_\alpha\}$. By the earlier observation, it suffices to find a maximum among $\{v(a+rx_\alpha+n) \mid r \in Z_p$ and $n \in N_\alpha\}$. Since N_α is nice in G, there is a $y \in N_\alpha$ so that $v(a+y) = v(a+N_\alpha)$. Then, for each r and n,

$$v(a+n) \leqslant v(a+y) = v(a+N_\alpha) < v(x_\alpha + N_\alpha) = v(x_\alpha) \leqslant v(rx_\alpha).$$

Therefore, $v(a+rx_\alpha+n) = v(a+n) \leqslant v(a+y)$, so that

$$v(a+y) = v(a + \langle X \rangle + N_\alpha).$$

Finally, set $M_\alpha = (\langle X \rangle + N_\alpha)/\langle X \rangle$. Since $\langle X \rangle + N_\alpha$ is nice in G, we know that M_α is nice in $G/\langle X \rangle$. Thus

$$0 = M_0 \subset \cdots \subset M_\lambda = G/\langle X \rangle$$

is a nice composition series for $G/\langle X \rangle$. Hence $G/\langle X \rangle$ is totally projective.

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