UNIFORM σ-ADDITIVITY IN SPACES OF BOCHNER OR PETTIS INTEGRABLE FUNCTIONS OVER A LOCALLY COMPACT GROUP

NICOLAE DINCULEANU

 Abstract. If \( G \) is an abelian locally compact group with Haar measure \( \mu \), \( E \) is a Banach space and \( K \subset L^1(G, \mu) \), we give necessary and sufficient conditions for the set \( \{ f, \ | f | d\mu : f \in K \} \) to be uniformly σ-additive in terms of uniform convergence on \( K \), for the topology \( \sigma(L^1, L^1_G) \) of convolution and translation operators. In case \( E = \mathbb{R} \), this gives a new characterization of relatively weakly compact sets \( K \subset L^1 \).

1. Introduction. In this paper we consider the space \( L^1_E \) of Bochner integrable functions and the space \( \mathcal{E}^1_E \) of Pettis integrable functions over an abelian locally compact group \( G \) endowed with a Haar measure \( \mu \), and we give a characterization of uniform σ-additivity in terms of uniform convergence— in the topology \( \sigma' = \sigma(L^1_E, \mathcal{E}^1_E) \), respectively in the weak topology of \( \mathcal{E}^1_E \)— of convolution and translation operators. If \( E = \mathbb{R} \), this yields a new characterization of relative weak compactness in \( L^1 \).

The convolution of Bochner integrable functions has been studied in [2] and has been extended in [4] for Pettis integrable functions.

Similar results have been obtained in a previous paper [3], where we give a characterization of uniform σ-additivity in the spaces \( L^1_E \) and \( \mathcal{E}^1_E \) over a measure space \((X, \Sigma, \mu)\), in terms of uniform convergence, in the \( \sigma' \)-topology or in the weak topology, of conditional expectations.

2. Uniform σ-additivity in the Lebesgue space \( L^1_E \). Let \( G \) be an abelian locally compact additive group endowed with a Haar measure \( \mu \); let \( E \) be a Banach space and \( L^1_E \) be the space of Bochner \( \mu \)-integrable functions \( f : G \to E \). For each relatively compact neighborhood \( V \) of 0 in \( G \), we choose a function \( u_V \) on \( G \) which is positive, bounded, symmetric (i.e. \( u_V(-t) = u_V(t) \)), vanishes outside \( V \) and \( \int u_V \, d\mu = 1 \). If \( \{V\} \) is a base of relatively compact neighborhoods of 0 in \( G \), we call \( \{u_V\}_{V \in \mathcal{V}} \) an approximate unit. We denote by \( u_V \ast f \) the convolution: \( u_V \ast f(t) = \int u_V(t - s)f(s) \, ds \), for \( t \in G \). For \( h \in G \) we denote by \( T^h \) the translation operator, defined by \( (T^h f)(t) = f(t + h) \), for \( t \in G \). If \( f \in L^1_E \), we denote by \( \int f \) the measure defined for any Borel set \( A \subset G \) by \( (\int f)(A) = \int_A f \, d\mu \). Finally, we denote by \( \sigma' \) the topology \( \sigma(L^1_E, \mathcal{E}^1_E) \) on...
If $E'$ has the Radon-Nikodym property, then the $\sigma'$-topology is the weak topology of $L^1_E$. 

**Theorem 1.** Let $K \subset L^1_E$ be a set.

I. The set $|K| \mu = \{|f| \mu; f \in K\}$ is uniformly $\sigma$-additive, if and only if the following conditions are satisfied:

(a) $K$ is bounded in $L^1_E$;
(b) For every countable subset $K_0 \subset K$, there exists a decreasing sequence $(V_n)$ of neighborhoods of 0 in $G$, such that either

- (b1) $\lim_n u_{V_n} \ast f = f$ in $L^1_F$ for the $\sigma'$-topology, uniformly for $f \in K_0$;
- (b2) $\lim_n T^h f = f$, in $L^1_F$, for the $\sigma'$-topology, uniformly for $f \in K_0$;

(c) $\lim_C \chi_C f = f$, strongly in $L^1_E$, uniformly for $f \in K$, where the limit is taken along the increasing net of all compact subsets of $G$.

(Condition (c) is superfluous if all functions of $K$ vanish outside a common compact set; in particular, if $G$ is compact.)

II. If $|K| \mu$ is uniformly $\sigma$-additive, then

- (b') $\lim_T u_{V} \ast f = f$ and
- (b'2) $\lim_{h \to 0} T^h f = f$, in $L^1_F$, for the $\sigma'$-topology, uniformly for $f \in K$.

**Proof.** Assume first conditions (a), (b1), and (c) satisfied and prove that $|K| \mu$ is uniformly $\sigma$-additive. Let $C \subset G$ be a compact set and $\phi \in L^1 \cap L^\infty$ a function with compact support $V$.

(A) The set $\phi \ast (\chi_C K) = \{\phi \ast (\chi_C f); f \in K\}$ is bounded in $L^1_E$. In fact, if $f \in K$, then

$$\|\phi \ast (\chi_C f)\|_1 \leq \|\phi\|_1 \|\chi_C f\|_1 \leq M \|\phi\|_1,$$

where $M = \sup\{\|f\|_1; f \in K\}$.

(B) The set $|\phi \ast (\chi_C K)| \mu$ is uniformly $\sigma$-additive. In fact the set $\phi \ast (\chi_C K)$ is bounded in $L^\infty_E$:

$$\|\phi \ast (\chi_C f)\|_\infty \leq \|\phi\|_\infty \|\chi_C f\|_1 \leq M \|\phi\|_1, \quad \text{for } f \in K.$$

It follows that the set $|\phi \ast (\chi_C K)| \mu$ is uniformly absolutely $\mu$-continuous. Since all the functions of $\phi \ast (\chi_C K)$ vanish outside the compact set $C + V$, the set $|\phi \ast (\chi_C K)| \mu$ is uniformly $\sigma$-additive.

(C) For every $g \in L^\infty_E$ and $f \in K$ we have

$$\left|\int \langle \phi \ast (\chi_C f) - f, g \rangle \, d\mu\right| \leq \left|\int \langle \phi \ast (\chi_C f - f), g \rangle \, d\mu\right| + \left|\int \langle \phi \ast f - f, g \rangle \, d\mu\right|$$

$$\leq \|\phi\|_1 \|\chi_C f - f\|_1 \|g\|_\infty + \left|\int \langle \phi \ast f - f, g \rangle \, d\mu\right|.$$

From condition (c) we deduce that there is an increasing sequence $(C_n)$ of compact sets such that $\lim_n \|\chi_{C_n} f - f\|_1 = 0$, uniformly for $f \in K$. Let $K_0 \subset K$ be a countable set. Taking above $C = C_n$ and $\phi = u_{V_n}$, where $(V_n)$ is the sequence stated in condition (b), we deduce that

$$\lim_n u_{V_n} \ast (\chi_{C_n} f) = f, \quad \text{in } L^1_E.$$
for the $\sigma'$-topology, uniformly for $f \in K_0$. Since, for each $n$, the set $\{u_n \ast (\chi_{c''} K_0) \} \setminus \mu$ is bounded and uniformly $\sigma$-additive, from Lemma 1b in [3] we deduce that $|K_0| \setminus \mu$ is also uniformly $\sigma$-additive. Since $K_0$ was an arbitrary countable set in $K$, it follows that $|K| \setminus \mu$ is uniformly $\sigma$-additive. If conditions (a), (b$_2$), and (c) are satisfied, then $|K| \setminus \mu$ is again uniformly $\sigma$-additive, since by Proposition 12 in [4], condition (b$_2$) implies (b$_1$). We remark that in [4], the implication $b_2 \Rightarrow b_1$ is stated for the weak topology, but the same proof is valid for the $\sigma'$-topology.

Conversely, assume $|K| \setminus \mu$ is uniformly $\sigma$-additive.

(D) $K$ is bounded in $L^1_{\mu}$. In fact, we can find a compact set $B \subseteq G$ such that $\int_{G - B} |f| \, d\mu \leq 1$ for all $f \in K_0$. Since $|K| \setminus \mu$ is uniformly absolutely $\mu$-continuous, there is $\eta > 0$ such that if $\mu(A) \leq \eta$, then $\int_{A} |f| \, d\mu \leq 1$ for all $f \in K$. Since the Haar measure is diffuse, it has the Darboux property: there is a finite family of disjoint Borel sets $A_1, \ldots, A_n$ with union $B$, such that $\mu(A_i) \leq \eta$ for $i = 1, \ldots, n$. It follows that $\int_{B} |f| \, d\mu \leq n$ for all $f \in K$, hence $\int |f| \, d\mu \leq n + 1$ for all $f \in K$: consequently $K$ is bounded.

(E) Since $|K| \setminus \mu$ is uniformly $\sigma$-additive, for every $\varepsilon > 0$ there is a compact set $\mathcal{C} \subseteq G$ such that

$$\int_{G - \mathcal{C}} |f| \, d\mu \leq \varepsilon, \quad \text{for all } f \in K,$$

that is

$$\int |\chi_{\mathcal{C}} f - f| \, d\mu \leq \varepsilon, \quad \text{for all } f \in K$$

and condition (c) follows.

(F) Let $f \in L^1_{\mu}$, $g \in L^\infty_{\mu}$, $\lambda > 0$, $h \in G$, and $C$ be an integrable subset. Then

$$\left| \int \langle T_h f - f, g \rangle \, d\mu \right| \leq 2\|g\|_{\infty} \int_{G - \mathcal{C}} |f| \, d\mu + 2\|g\|_{\infty} \int_{\{f \sim \lambda\}} |f| \, d\mu + \lambda \|T^{-h}(\chi_{c'} g) - \chi_{c'} g\|_1 + \lambda \|g\|_{\infty} \|\chi_{c'} - \chi_{c''}\|_1.$$

In fact

$$\left| \int \langle T_h f - f, g \rangle \, d\mu \right| \leq \left| \int \langle T_h (f \chi_{G - c'}) - f\chi_{G - c'}, g \rangle \, d\mu \right| + \left| \int \langle T_h (f\chi_{c'}) - f\chi_{c''}, g \rangle \, d\mu \right|.$$

The first term can be written

$$\left| \int \langle T_h (f\chi_{G - c'}) - f\chi_{G - c'}, g \rangle \, d\mu \right| = \left| \int \langle f\chi_{G - c'}, T^{-h} g - g \rangle \, d\mu \right| \leq 2\|g\|_{\infty} \int_{G - \mathcal{C}} |f| \, d\mu.$$
For the second term we have $T^h(f \chi_c) = \chi_{c-h} T^h(f \chi_c)$, hence

$$
\left| \int \langle T^h(f \chi_c) - f \chi_c, g \rangle \, d\mu \right| = \left| \int \langle f \chi_c, T^{-h}(g \chi_{c-h}) - g \chi_c \rangle \, d\mu \right|
$$

$$
\leq \left| \int_{c \cap \{|y| > \lambda\}} \langle f, T^{-h}(g \chi_{c-h}) - g \chi_c \rangle \, d\mu \right|
$$

$$
+ \left| \int_{c \cap \{|y| < \lambda\}} \langle f, T^{-h}(g \chi_{c-h}) - g \chi_c \rangle \, d\mu \right|
$$

$$
\leq 2\|g\|_\infty \int_{\{|y| > \lambda\}} |f| \, d\mu + \lambda \|T^{-h}(g \chi_{c-h}) - g \chi_c \|_1
$$

$$
\leq 2\|g\|_\infty \int_{\{|y| > \lambda\}} |f| \, d\mu + \lambda \|T^{-h}(g \chi_c) - g \chi_c \|_1
$$

$$
+ \lambda \|T^{-h}g(\chi_{c-h} - \chi_c)\|_1
$$

and this last term is smaller than $\lambda \|g\|_\infty \|\chi_{c-h} - \chi_c\|_1$.

(G) We can now prove conditions (b2) and (b\',). Let $g \in L^\infty_E$, $g \neq 0$ and $\epsilon > 0$. Take $C \subset G$ such that

$$
\int_{G-C} |f| \, d\mu < \epsilon / (8\|g\|_\infty), \quad \text{for all } f \in K.
$$

Take also $\lambda > 0$ such that

$$
\int_{\{|y| > \lambda\}} |f| \, d\mu < \epsilon / (8\|g\|_\infty), \quad \text{for all } f \in K.
$$

We can find a symmetric neighborhood $V$ of 0 such that for all $h \in V$ we have

$$
\|T^{-h}(\chi_c g) - \chi_c g\|_1 < \epsilon / 4,
$$

and

$$
\|\chi_c - \chi_{c-h}\|_1 = \|\chi_c - T^{-h}\chi_c\|_1 < \epsilon / (4\lambda \|g\|_\infty).
$$

Then, for $h \in V$ and all $f \in K$ we have, from step (F),

$$
\left| \int \langle T^h f, g \rangle \, d\mu \right| < \epsilon;
$$

that is $\lim_{h \to 0} T^h f = f$, in $L^1_E$ for the $\sigma'$-topology, uniformly for $f \in K$. This proves condition (b2); and condition (b\') follows from Proposition 12 in [3].

(H) To prove condition (b2), let $K_0 \subset K$ be a countable subset. The proof of condition (b1) is the same as in step (G).

Let $R_0$ be a countable ring of relatively compact Borel subsets of $G$, such that any function of $K_0$ is the limit $\mu$-a.e. and in $L^1_E$ of step functions over $R_0$.

Since for each $f \in L^1_E$ we have $\lim_{h \to 0} T^h f = f$, strongly in $L^1_E$, we can find a decreasing sequence $(V_n)$ of symmetric neighborhoods of 0, such that

$$
\lim_{h \in V_n} T^h \chi_A = \chi_A, \quad \text{strongly in } L^1 \text{ for every } A \in R_0.
$$

Next, we choose arbitrarily a sequence $(h_n)$ such that $h_{2n-1} = -h_{2n} \in V_n$ for every $n$. Then $\lim_{n} T^{h_n} \chi_A = \chi_A$ strongly in $L^1$ for every $A \in R_0$. Consider the group $\Gamma \subset G$ generated by the
sequence \((h_n)\). Then the set \(L\) of linear combinations of functions of the form 
\((T^{\alpha_1} \chi_{A_1})(T^{\alpha_2} \chi_{A_2}) \cdots (T^{\alpha_k} \chi_{A_k})\) with \(\alpha_1, \ldots, \alpha_k \in \Gamma\) and \(A_1, \ldots, A_k \in R_0\), is an algebra of \(\mu\)-integrable functions, invariant with respect to \(T^{\alpha}\) for any \(\alpha \in \Gamma\). Moreover, \(\lim_n T^{h_n} \chi = \chi\), in \(L^1\), for all \(\chi \in L\).

It is enough to check this for the functions of the form \(\chi = (T^{\alpha} \chi_{A_1})(T^{\alpha_2} \chi_{A_2})\). We have \(\lim_n T^{h_n} \chi = \chi, \text{ } \mu\text{-a.e. and since } |T^{h_n} \chi| \leq \chi_{V_1} + (A_1 \cup A_2)\), we can apply Lebesgue’s dominated convergence theorem and deduce that \(\lim_n T^{h_n} \chi = \chi\) in \(L^1\).

Moreover, this last equality remains valid for \(\chi\) in the closure of \(L\) in \(L^1\), since \(\sup_n \|T^{h_n}\| = 1\).

The class \(\Lambda = \{A; \chi_A \in L\}\) is a ring containing \(R_0\), and the class \(\{\chi_A; A \in \Lambda\}\) is invariant with respect to \(T^{\alpha}\) for all \(\alpha \in \Gamma\). All functions of \(L\) vanish \(\mu\)-a.e. outside a \(\sigma\)-finite set \(X_0\).

The \(\delta\)-ring \(\Sigma_0\) generated by \(\Lambda\) is the completion of \(\Lambda\) for the semidistance \(\rho(A, B) = \mu(A \Delta B) = \|\chi_A - \chi_B\|_1\), and can be obtained—modulo negligible sets—as closure in \(L^1\) of the set of functions \(\chi_A\) with \(A \in \Lambda\). It follows that the class \(\{\chi_A; A \in \Sigma_0\}\) is invariant with respect to \(T^{\alpha}\) for all \(\alpha \in \Gamma\), since the class \(\{\chi_A; A \in \Lambda\}\) has this property, and since this property is preserved by passing to limits in \(L^1\).

We deduce then that for any Banach space \(F\), the space \(L^1_E(X_0, \Sigma_0, \mu)\) is invariant with respect to \(T^{\alpha}\) for all \(\alpha \in \Gamma\), and that for every \(f \in L^1_E(X_0, \mu)\) we have \(\lim_n T^{h_n} f = f\), strongly in \(L^1_E(X_0, \mu)\).

In fact this property is valid for all step functions, and \(\sup_n \|T^{h_n}\| = 1\).

We are now ready to prove condition (b_2).

Let \(g \in L^\infty_E, g \neq 0, \text{ and } \varepsilon > 0\). The conditional expectation \(g' = E(g \mid \Sigma_0)\) is defined since the space \((G, \mu)\) is localizable (see [1]).

We can consider \(g' \in L^\infty_E(X_0, \Sigma_0, \mu)\). Since \(K_0 \subset L^1_E(X_0, \Sigma_0, \mu)\) and \(|K_0| \mu\) is uniformly \(\sigma\)-additive, there is a set \(C \in \Sigma_0\) such that

\[
\int_{G-C} |f| \, d\mu < \varepsilon / (8 \|g\|_\infty), \quad \text{for all } f \in K_0.
\]

Also let \(\lambda\) be such that

\[
\int_{\{|\mu| > \lambda\}} |f| \, d\mu < \varepsilon / (8 \|g\|_\infty), \quad \text{for all } f \in K_0.
\]

Since \(\chi_C g' \in L^1_E(X_0, \Sigma_0, \mu)\), we have \(\lim_n T^{h_n}(\chi_C g') = \chi_C g'\), strongly in \(L^1_E\); and we have also

\[
\lim_n \|\chi_C - \chi_{C-h_n}\|_1 = \lim_n \|\chi_C - T^{h_n} \chi_C\|_1 = 0.
\]

Let \(n_\varepsilon\) be such that for \(n \geq n_\varepsilon\) we have

\[
\|T^{h_n}(\chi_C g') - \chi_C g'\|_1 < \varepsilon / (4 \lambda)
\]

and

\[
\|\chi_C - \chi_{C-h_n}\|_1 < \varepsilon / (4 \lambda \|g\|_\infty)
\]
Then, for any \( f \in K_0 \) and any \( n \geq n_0 \), we deduce from step (F),

\[
\left| \int \langle T^h f, g \rangle \, d\mu \right| = \int \langle T^h f, g' \rangle \, d\mu < \varepsilon.
\]

that is \( \lim_n T^h f = f \) in \( L^1_E \) for the \( \sigma' \)-topology, uniformly for \( f \in K_0 \). Since the sequence \( (h_n) \) was arbitrary, it follows that

\[
\lim_{h \in V_n, n \to \infty} T^h f = f, \quad \text{in } L^1_E
\]

for the \( \sigma' \)-topology, uniformly for \( f \in K \) and so, condition (b_1) is proved. Condition (b_2) then follows from Proposition 12 in [4]; and this completes the proof of the theorem.

**Remark.** The \( \sigma' \)-topology cannot be replaced by the weak topology. There are examples (to be published in a joint paper with Jürgen Batt) of relatively weakly compact sets \( K \subset L^1_E \) over the circle group, such that the limits in (b_1) and (b_2) for the weak topology are false.

3. **Uniform \( \sigma \)-additivity in the Pettis space \( L^1_E \).** We denote by \( L^1_E \) the Pettis space of functions \( f: G \to E \) which are strongly \( \mu \)-measurable and Pettis integrable, endowed with the Pettis norm

\[
(f)_1 = \sup \left\{ \int |\langle f, x' \rangle| \, d\mu ; \, x' \in E_1^* \right\}
\]

where \( E_1^* \) is the unit ball of \( E^* \). A set \( F \subset E_1^* \) is norming for a set \( K \subset L^1_E \), if

\[
|f(t)| = \sup \{ |\langle f(t), x' \rangle| ; \, x' \in F \}, \mu\text{-a.e. for every } f \in K.
\]

**Theorem 2.** Let \( K \subset L^1_E \) be a set.

1. The set \( K \mu \) is uniformly \( \sigma \)-additive, if and only if the following conditions are satisfied:

   (a) \( K \) is bounded in \( L^1_E \);

   (b) For every countable subset \( K_0 \subset K \) there is a decreasing sequence \( (V_n) \) of neighborhoods of 0 and a countable subset \( E_0^* \subset E_1^* \) norming for \( K_0 \), such that either

   (b_1) \( \lim_n \langle u_{V_n} \ast f, x' \rangle = \langle f, x' \rangle \), weakly in \( L^1 \), uniformly for \( f \in K_0 \) and \( x' \in E_0^* \);

   or

   (b_2) \( \lim_{h \in V_n, n \to \infty} \langle T^h f, x' \rangle = \langle f, x' \rangle \), weakly in \( L^1 \), uniformly for \( f \in K_0 \) and \( x' \in E_0^* \);

   (c) \( \lim_{C \to \infty} f_k \chi_C = f \), strongly in \( L^1_E \), uniformly for \( f \in K \), the limit being taken along the increasing net of all compact subsets of \( G \).

2. If \( K \mu \) is uniformly \( \sigma \)-additive, then:

   (b_1') \( \lim_{v} \langle u_v \ast f, x' \rangle = \langle f, x' \rangle \) and

   (b_2') \( \lim_{h \to \infty} \langle T^h f, x' \rangle = \langle f, x' \rangle \)

weakly in \( L^1 \), uniformly for \( f \in K \) and \( x' \in E_1^* \).

**Proof.** Assume first conditions (a), (b) and (c) satisfied. Let \( K_0 \subset K \) be a countable set, and \( E_0^* \) the set corresponding to \( K_0 \) by condition (b) above.

Then the set \( \{ \langle f, x' \rangle ; \, f \in K_0, x' \in E_0^* \} \) is countable and satisfies conditions (a), (b) and (c) of Theorem 1, in the space \( L^1 \). It follows that \( \langle K_0, E_0^* \rangle \mu \) is
uniformly $\sigma$-additive; and then $K_0\mu$ is also uniformly $\sigma$-additive; therefore $K\mu$ is uniformly $\sigma$-additive.

Conversely, assume $K\mu$ is uniformly $\sigma$-additive; let $K_0 \subseteq K$ be countable, and let $E'_0 \subseteq E'_1$ be a countable set, norming for $K_0$. The set $\langle K_0, E'_0 \rangle$ is countable and uniformly $\sigma$-additive. From Theorem 1 we deduce:

(a) The set $\langle K_0, E'_0 \rangle$ is bounded in $L^1$; hence $K_0$ is bounded in $L^1$; therefore $K$ is bounded in $L^1$;

(b) There exists a decreasing sequence $(V_n)$ of neighborhoods of 0, satisfying conditions (b1) and (b2) of this theorem;

(c) $\lim_{r} \langle \chi_{E'_0} f, x' \rangle = \langle f, x' \rangle$, strongly in $L^1$ uniformly for $f \in K$ and $x' \in E'_0$, which is equivalent to condition (c) of this theorem.

Finally, to obtain (b1) and (b2) we apply the second part of Theorem 1 to the set $\langle K, E'_0 \rangle$.

Note. We take this opportunity to mention that Theorems 2(iii) and 4(iii) in [3] are valid without the condition $\sup \{|f(t)| : t \in K \} < \infty$ $\mu$-a.e. The proof will be given in a forthcoming paper, for a more general situation.

References


Department of Mathematics, University of Florida, Gainesville, Florida 32611