$F_p$ CLASSES AND HYPERGEOMETRIC SERIES

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ABSTRACT. For $\Re(p) > 0$, let $F_p = \{ f : f(z) = \int (1 - xz)^{-p} d\mu(x), |z| < 1 , \mu$ a probability measure on $|x| = 1 \}$. Let $F_p', F_q' = \{ f \cdot g : f \in F_p, g \in F_q \}$. Brickman, Hallenbeck, MacGregor and Wilken proved that $p > 0$ and $q > 0$, then $F_p \subseteq F_{p+q}$. Kent Pearce recently proved a converse result: if $F_p \subseteq F_{p+q}$, then $p > 0$ and $q > 0$, or $p = q = 1 + it$ for some real $t$. The case $p = q = 1 + it$, $t \neq 0$, will be excluded. Consequently a full converse of the above theorem holds.

Introduction. Let $F_p = \{ f : f(z) = \int_{|x|=1} (1 - xz)^{-p} d\mu(x), |z| < 1, \mu$ a probability measure on $|x| = 1 \}$, where $\Re p > 0$. Also let $F_p \cdot F_q = \{ f \cdot g : f \in F_p , g \in F_q \}$, where $\cdot$ denotes the Hadamard convolution. Pearce [1] proved that if $F_p \cdot F_q \subseteq F_{p+q}$ then $p > 0$ and $q > 0$, or $p = q = 1 + it$ for some real $t$. In this paper, the case $p = q = 1 + it$, $t \neq 0$, will be excluded, using an identity involving hypergeometric functions.

Pierce [1] used the following theorem in establishing his result.

Theorem A. If $F_p \cdot F_q \subseteq F_{p+q}$ then

$$\Re \left( \frac{1}{1 - xz} - 2F1 \left( \frac{1}{p + q} : \frac{(y - x)z}{1 - xz} \right) \right) > \frac{1}{2}$$

for all $x, y$ and $z$ such that $|x| = |y| = 1$ and $|z| < 1$.

Here the hypergeometric function $2F1(a,b;c;z)$ is defined for $|z| < 1$ by

$$2F1 \left( \begin{array}{c} a, b \\ c \end{array} ; z \right) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k,$$

where $(a)_n = a(a+1)(a+2) \cdots (a+n-1), n \geq 0$, and $(a)_0 = 1$.

If $\Re(c) > \Re(b) > 0$, then the $2F1$ has an integral representation

$$2F1 \left( \begin{array}{c} a, b \\ c \end{array} ; z \right) = \frac{\Gamma(c)}{\Gamma(c - b) \Gamma(b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} dt.$$  

This provides an analytic continuation of the $2F1$ to $\mathbb{C} - [1, \infty)$ under the restriction $\Re(c) > \Re(b) > 0$. Using some recurrence relations involving the parameters, the $2F1$ can be analytically continued to $\mathbb{C} - [1, \infty)$ for all $a, b, c$ such that $c \neq 0, -1, \ldots$ (see Slater [3]).
Also we note the simple identity

\[ _2F_1\left( \begin{array}{c} a, b \\ c \end{array} : z \right) = (1 - z)^{-b}, \quad a \neq 0, -1, \ldots. \]

**The main result.**

**Theorem 1.** If \( F_{1+t} F_{1+t} \subseteq F_{2+t}, \) \( t \) real, then \( t = 0. \)

**Proof.** By Theorem A we must have

\[ \Re \left( \frac{1}{1 - xz} _2F_1\left( \begin{array}{c} 1, 1 + it \\ 2 + 2it \end{array} : \frac{(y - x)z}{1 - xz} \right) \right) > \frac{1}{2}, \quad \text{with} \quad |x| = |y| = 1, |z| < 1. \]

We will assume \( t \neq 0 \) and obtain a contradiction. First we state a lemma proved by Slater [3, p. 34],

\[ _2F_1\left( \begin{array}{c} a, b \\ c \end{array} : z \right) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} _2F_1\left( \begin{array}{c} a, b \\ a + b - c + 1 \end{array} : 1 - z \right) + (1 - z)^{a - b - c} \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)} _2F_1\left( \begin{array}{c} c - a, c - b \\ c - a - b + 1 \end{array} : 1 - z \right). \]

\( c \neq 0, -1, -2, \ldots, a + b - c \neq 0, \pm 1, \pm 2, \ldots, |\text{Arg}(z)| < \pi, |\text{Arg}(1 - z)| < \pi. \)

The lemma is useful in that it reveals the behavior of \( _2F_1 \) when \( z \to 1 \) with rather relaxed restrictions on the parameters \( a, b \) and \( c \). It is clear from (6) that if \( \Re(c - a - b) > 0 \) then

\[ _2F_1\left( \begin{array}{c} a, b \\ c \end{array} : z \right) \to \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} \quad \text{as} \quad z \to 1. \]

If \( \Re(c - a - b) < 0 \) then the \( _2F_1(\ldots) \) goes to infinity like \( (1 - z)^{c - a - b} \) as \( z \to 1. \)

The case where \( \Re(c - a - b) = 0 \) can also be treated by (6) and that is what will make the proof of this theorem work.

Now let \( z = r e^{i\theta}, 0 < r < 1, \) and \( xy = \xi \neq 1. \) Then by (6),

\[ _2F_1\left( \begin{array}{c} 1, 1 + it \\ \frac{1 - \xi}{1 - \xi} \end{array} : \frac{(y - x)z}{1 - xz} \right) = _2F_1\left( \begin{array}{c} 1, 1 + it \\ 2 + 2it \end{array} : \frac{r(1 - \xi)}{1 - \xi r} \right) = \frac{\Gamma(2 + 2it)\Gamma(it)}{\Gamma(1 + 2it)\Gamma(1 + it)} _2F_1\left( \begin{array}{c} 1, 1 + it \\ 1 - it \end{array} : \frac{1 - r}{1 - \xi r} \right) + \left( \frac{1 - r}{1 - \xi} \right)^{it} \frac{\Gamma(2 + 2it)\Gamma(-it)}{\Gamma(1)\Gamma(1 + it)} _2F_1\left( \begin{array}{c} 1 + 2it, 1 + it \\ 1 + it \end{array} : \frac{1 - r}{1 - \xi r} \right). \]

Here the powers are computed using the principal value of the arguments, for this was assumed in (6). In order to simplify (7) we use (4) and the following identities [2, pp. 12–24].

\[ (i) \quad \Gamma(z + 1) = z\Gamma(z), \quad z = 0, -1, -2, \ldots. \]

\[ (ii) \quad \sqrt{\pi} \Gamma(2z) = 2^{2z - 1} \Gamma(z), \quad z \neq 0, -\frac{1}{2}, -1, -\frac{3}{2}, \ldots. \]

\[ (iii) \quad \Gamma(1/2) = \sqrt{\pi}. \]
We then obtain

\[(9) \quad \frac{1}{1 - xz} F_1 \left( \begin{array}{c} 1, 1 + it, \frac{(y - x)z}{2 + 2it} \\ 1 - \xi \end{array} \right) = \frac{2 - i/t}{1 - \xi} F_1 \left( \begin{array}{c} 1, 1 + it, 1 - r \\ 1 - it, 1 - \xi \end{array} \right) \]

\[+ (1 - r)^{n/2} \Gamma(\frac{3}{2} + it) \Gamma(-it) \left( \frac{1}{2} (1 - \xi) r \right)^{-1} \left( 1 - \xi^2 \right) \cdot \]

\[\xi = x\zeta, \quad z = r\zeta, \quad 0 < r < 1. \] The right side of (9) has the form

\[(10) \quad A(r, \xi) + (1 - r)^{n/2} B(r, \xi), \]

where

\[\lim_{r \to 1} A(r, \xi) = \frac{2 - i/t}{1 - \xi} \]

and

\[\lim_{r \to 1} B(r, \xi) = \left( \frac{1}{2} (1 - \xi) \right)^{-1} (1 - \xi)^{n/2} \Gamma(\frac{3}{2} + it) \Gamma(-it). \]

Note that as \( r \to 1^+ \), \((1 - r)^{n/2} = \exp(it \ln(1 - r))\) is a complex number which revolves around the origin more and more rapidly while the modulus is one. Consequently, we must have

\[(12) \quad \Re A(r, \xi) > \frac{1}{2}, \quad 0 < r < 1, |\xi| = 1, \xi \neq 1. \]

Letting \( r \to 1^+ \), we have

\[(13) \quad \Re \left( \frac{2 - i/t}{1 - \xi} \right) > \frac{1}{2}. \]

But if we let \( \xi = e^{i\theta} \)

\[\Re \left( \frac{2 - i/t}{1 - \xi} \right) = 1 - \frac{(1 + \cos \theta)}{2t \sin \theta} \]

and it is clear that a nonzero value of \( \theta \) can be chosen so that the above expression is less than \( \frac{1}{2} \). We have obtained a contradiction and the theorem is proved. \( \square \)

An obvious consequence of this theorem is

**Corollary.** If \( F_p \cdot F_q \subseteq F_p \cdot q \) then \( p > 0 \) and \( q > 0 \).

**References**


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