FOURIER COEFFICIENTS OF CONTINUOUS FUNCTIONS ON COMPACT GROUPS

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Abstract. Let G be an infinite compact group with dual object Σ. Letting ℌσ be the representation space for σ ∈ Σ, ℌ2(Σ) is the set \( \{ A = (A^σ) ∈ \prod \mathcal{B}(\mathcal{H}_σ) : \| A \|_2 = \sum_σ d_σ \text{Tr}(A^σA^{σ'}) < ∞\} \). For \( A ∈ ℌ^2(Σ) \), we show that there is a function \( f \) in \( C(G) \) such that \( \| f \|_∞ ≤ C \| A \|_2 \) and \( \text{Tr}(f(σ)f(σ)^*) ≥ \text{Tr}(A^σA^{σ'}) \) for every \( σ ∈ Σ \).

In a 1977 paper [3], K. de Leeuw, Y. Katznelson and J.-P. Kahane proved that every square summable sequence is dominated by the sequence of Fourier coefficients of a continuous function on the circle group, \( T \). As the authors mentioned, this result is true, with the same proof, for any compact abelian group in the role of \( T \) and its dual group in place of the integers, \( Z \). This paper answers the same question for a compact nonabelian group. Using appropriate tools, our proof parallels that of [3]. All notation and terminology used here without explicit definition is as in [2].

Let \( G \) be an infinite compact group with dual object Σ. For each \( σ ∈ Σ \), let \( U^σ \) be a representation in \( σ \) and let \( \mathcal{H}_σ \), its representation space, have dimension \( d_σ \). If \( \mathcal{B}(\mathcal{H}_σ) \) is the space of operators on \( \mathcal{H}_σ \), define \( \| \cdot \|_2 \) on \( \mathcal{B}(\mathcal{H}_σ) \) by \( \| A^σ \|_2 = \text{Tr}(A^σA^{σ'})^{1/2} \). Let \( ℌ(Σ) = \prod_σ \mathcal{B}(\mathcal{H}_σ) \) and let \( ℌ^2(Σ) \) be the set of \( A = (A^σ) ∈ ℌ(Σ) \) satisfying

\[ \| A \|_2 = \left( \sum_σ d_σ \| A^σ \|_2^2 \right)^{1/2} < ∞. \]

Finally, \( \Gamma \) will designate the compact group \( \prod_σ \mathcal{U}(d_σ) \), where \( \mathcal{U}(d_σ) \) is the group of all unitary operators on \( \mathcal{H}_σ \).

We make use of the following results.

1. Let \( f(V) = \sum_σ d_σ \text{Tr}(B^σV^σ) \) \( (V ∈ Γ) \) be a finite sum. Then

\[ \int_Γ |\exp(f(V))|dV ≤ \exp(\| B \|_2^2). \]

This statement and its proof are similar to [4, Lemma 2].

2. Suppose \( A ∈ ℌ^2(Σ) \). Then, for almost all \( V ∈ Γ \),

\[ \sum_σ d_σ \text{Tr}(A^σV^σU^σ(x)) \]

converges for almost every \( x ∈ G \) [4, Lemma 8].

Received by the editors May 24, 1982.
1980 Mathematics Subject Classification. Primary 43A30, 43A77, 60B15.
Lemma 1. Suppose \( A \in \mathcal{S}^2(\Sigma) \). For every \( V \in \Gamma \), let \( f_V \) be the function in \( L^2(G) \) whose Fourier series is
\[
\sum_\sigma d_\sigma \text{Tr}(A^\sigma V^\sigma U^\sigma(x)).
\]
Let \( \lambda > 0 \). Then the set
\[
E = \left\{ V \in \Gamma : \int_G \exp(\lambda|f_V(x)|)dx \leq 2 \exp\left(4\lambda^2\|A\|_2^2\right) \right\}
\]
has positive \( \Gamma \)-measure.

Proof. Let \( x \in G \) be fixed but arbitrary. Let \( S_n f_V \) be the \( n \)th partial sum of \( f_V \) and \( S_n \), that of \( \|A\|_2^2 \). (These partial sums are relative to some arbitrary and fixed enumeration of the set \( \{ \sigma \in \Sigma : A^\sigma \neq 0 \} \).)

Using the inequality \( \exp|z| \leq \sum_{k=1}^\infty \frac{|z|^k}{k!} \) the invariance of the Haar integral on \( \Gamma \) and (1),
\[
\int_\Gamma \exp(\lambda|S_n f_V(x)|)dV \leq 2 \exp(4\lambda^2 S_n^2).
\]
As \( x \in G \) was arbitrary,
\[
\int_\Gamma \int_G \exp(\lambda|S_n f_V(x)|)dV dx \leq 2 \exp(4\lambda^2 S_n^2).
\]
By Fubini's theorem, the order of integration can be reversed. Applying (2) and Fatou's lemma,
\[
\int_\Gamma \int_G \exp(\lambda|f_V(x)|)dx dV \leq 2 \exp(4\lambda^2\|A\|_2^2).
\]
Consequently, \( E \) has positive \( \Gamma \)-measure.

Lemma 2. Suppose \( B \in \mathcal{S}^2(\Sigma) \). Let \( \varepsilon, \eta > 0 \) be given and suppose that \( \|B\|_2 \leq \varepsilon \).
Then there is a choice of \( V \in \Gamma \) such that the function \( h \in L^2(G) \), whose Fourier series is \( \sum_\sigma d_\sigma \text{Tr}(B^\sigma V^\sigma U^\sigma(x)) \), satisfies
\[
\left\| \left( |h| - \eta \right)^+ \right\|_2 \leq 16 \sqrt{2} e^{-\frac{\eta^2}{8\varepsilon^2}} \exp\left(\frac{-\eta^2}{32\varepsilon^2}\right).
\]

Proof. Let \( \lambda = \eta/8\varepsilon^2 \). Invoke Lemma 1 to obtain \( V \in \Gamma \) so that
\[
\int_G \exp(\lambda|h(x)|)dx \leq 2 \exp(4\lambda^2\|B\|_2^2) \leq 2 \exp\left(\frac{\eta^2}{16\varepsilon^2}\right).
\]
Since
\[
\sup_{t \geq \eta} (t - \eta)^2 \exp(-\lambda t) = 256 e^{-2\varepsilon^2} \eta^{-2} \exp\left(\frac{-\eta^2}{8\varepsilon^2}\right).
\]

\[
\left\| \left( |h| - \eta \right)^+ \right\|_2 \leq 256 e^{-2\varepsilon^2} \eta^{-2} \exp\left(\frac{-\eta^2}{8\varepsilon^2}\right) \int_G \exp(\lambda|h(x)|)dx
\leq 512 e^{-2\varepsilon^2} \eta^{-2} \exp\left(\frac{-\eta^2}{16\varepsilon^2}\right).
\]
Lemma 3. Suppose $A \in \mathcal{D}^2(\Sigma)$. There exists a function $f \in L^\infty(G)$ with $\|f\|_\infty \leq 36\|A\|_2$, and $\|f(\sigma)\|_2 \geq \|A^\sigma\|_2$ for every $\sigma \in \Sigma$.

Proof. Assume that $\|A\|_2 = 1$. Define sequences $(\delta_j)_{j=1}^\infty$, $(\eta_j)_{j=0}^\infty$, and $(\varepsilon_j)_{j=0}^\infty$ by

\[ \delta_j = 3^{-j}; \quad \eta_j = 36\delta_j; \quad \varepsilon_0 = 1 \]

and

\[ \varepsilon_{j+1} = 32\sqrt{2}(1 - \delta_j) e^{-1} \varepsilon_j^2 \eta_{j+1}^{-1} \delta_j^{-1} \exp \left( \frac{-\eta_{j+1}^2}{32\varepsilon_j^2} \right) \quad \text{for } j \geq 0. \]

One checks by induction that $\varepsilon_j \leq 6^{-j}$ for $j \geq 0$. Let $s_0 = 0$ and $s_k = \sum_{j=1}^k \delta_j$.

We next define sequences of functions $(f_j)_{j=0}^\infty$, $(g_j)_{j=0}^\infty$, and $(h_j)_{j=0}^\infty$ with $f_j = g_j + h_j$, which satisfy

(a) $\|g_j\|_\infty \leq 36s_j$;
(b) $\|h_j\|_2 \leq \varepsilon_j$;
(c) $\left\| \left( |h_j| - \eta_{j+1} \right)^+ \right\|_2 \leq \rho_j = 16\sqrt{2} e^{-1} \varepsilon_j^2 \eta_{j+1}^4 \exp \left( \frac{-\eta_{j+1}^2}{32\varepsilon_j^2} \right)$;
(d) $\left\| \hat{f}_j(\sigma) \right\|_2 \geq (1 - s_j)\|A^\sigma\|_2$ for $\sigma \in \Sigma$.

Let $g_0 = 0$ and choose $h_0$ to satisfy the conclusion of Lemma 2 with $B = A$, $\varepsilon = \varepsilon_0$ and $\eta = \eta_1$. Then (a)–(d) are true for $j = 0$.

Suppose that $k \geq 1$ and that $g_{k-1}$ and $h_{k-1}$ have been selected to satisfy (a)–(d) when $j = k - 1$. Define

\[ g_k(x) = \begin{cases} 36s_k \text{sgn} f_{k-1}(x) & \text{if } |f_{k-1}(x)| > 36s_k, \\ f_{k-1}(x) & \text{if } |f_{k-1}(x)| \leq 36s_k. \end{cases} \]

Then (a) holds for $j = k$. Note that if $|f_{k-1}(x)| > 36s_k$, then

\[ |f_{k-1}(x) - g_k(x)| = |f_{k-1}(x)| - 36s_k \leq |g_{k-1}(x)| - 36s_{k-1} + |h_{k-1}(x)| - \eta_k \leq (|h_{k-1}(x)| - \eta_k)^+. \]

Thus $\|f_{k-1} - g_k\|_2 \leq \rho_{k-1}$. Let

\[ \Phi = \{ \sigma \in \Sigma: \|\hat{g}_k(\sigma)\|_2 < (1 - s_k)\|A^\sigma\|_2 \}. \]

For $\sigma \in \Phi$, we have $\|\hat{f}_{k-1}(\sigma) - \hat{g}_k(\sigma)\|_2 \geq \delta_k \|A^\sigma\|_2$. Define $B^\sigma \in \mathcal{D}^2(\Sigma)$ by

\[ B^\sigma = \begin{cases} 2(1 - s_k)A^\sigma & \text{for } \sigma \in \Phi, \\ 0 & \text{otherwise}. \end{cases} \]
Then,
\[ \| B \|_2^2 = \sum_{\sigma \in \Phi} d_\sigma A(1 - s_\sigma)^2 \| A^\sigma \|_2^2 \]
\[ \leq 4(1 - s_\sigma)^2 \| f_{k - 1} - g_k \|_2^2 \]
\[ \leq 4(1 - \delta_\sigma)^2 \| \rho_{k - 1}^2 \| = \epsilon_k^2. \]
Thus a function \( h_k \) can be chosen via Lemma 2 applied to \( B, \epsilon = \epsilon_k \) and \( \eta = \eta_k + 1 \) which satisfies both (b) and (c). Finally, if \( \sigma \in \Phi \),
\[ \| f_\sigma \|_2 = \| g_\sigma \|_2 \geq (1 - s_\sigma) \| A^\sigma \|_2, \]
while if \( \sigma \notin \Phi \),
\[ \| f_\sigma \|_2 \geq \| B^\sigma \|_2 - \| g_\sigma \|_2 \geq (1 - s_\sigma) \| A^\sigma \|_2. \]
Therefore (d) holds for \( j = k \). This completes the definition of these sequences of functions.

For \( j \geq 1 \),
\[ \| f_{j - 1} - f_j \|_2 \leq \| f_{j - 1} - g_j \|_2 + \| h_j \|_2 < 2(6^{-j}). \]
so there is a function \( f \in L^2(G) \) such that \( \lim_{j \to \infty} \| 2f_j - f \|_2 = 0 \). Since \( \| 2g_j - f \|_2 \leq \| 2f - f \|_2 + 2\| h \|_2 \),
\[ \lim_{j \to \infty} \| 2g_j - f \|_2 = 0. \]
Thus a subsequence of \( (2g_j)_{j=0}^\infty \) converges to \( f \) pointwise almost everywhere on \( G \).

Hence
\[ \| f \|_\infty \leq 2 \lim_{j \to \infty} \| g_j \|_\infty \leq 36 \]
and, for each \( \sigma \in \Sigma \),
\[ \| \hat{f}(\sigma) \|_2 \geq 2 \lim_{j \to \infty} (1 - s_j) \| A^\sigma \|_2 = \| A^\sigma \|_2. \]
This verifies the lemma.

We are now able to state and prove the main result, which replaces \( L^\infty(G) \) in Lemma 3 by \( C(G) \).

**Theorem.** Suppose \( A \in D^2(\Sigma) \). There exists a function \( f \in C(G) \) with \( \| f \|_\infty \leq 37 \| A \|_2 \) and \( \| \hat{f}(\sigma) \|_2 \geq \| A^\sigma \|_2 \) for every \( \sigma \in \Sigma \).

**Proof.** Let \( \delta = \| A \|_2/36 \). Assume \( \delta > 0 \). Let \( h \in L^2(G) \) have Fourier series
\[ \sum_{\sigma} d_\sigma \text{Tr}(A^\sigma U^\sigma(X)). \]
By a factorization theorem, there exist functions \( g \in L^2(G) \) and \( k \in L^1(G) \) such that
\[ h = k \ast g; \]
\( k \) is nonnegative and central in \( L^1(G) \);
\[ \| k \|_1 = 1; \]
\[ \| h - g \|_2 < \delta. \]
(See [2, (32.31)], replacing \( C(G) \) by \( L^2(G) \).)
Invoke Lemma 3 to obtain a function \( f_\infty \in L^\infty(G) \) which satisfies \( \|f_\infty\|_\infty \leq 36\|g\|_2 \) and \( \|\hat{f}_\infty(\sigma)\|_2 \geq \|\hat{g}(\sigma)\|_2 \) for every \( \sigma \in \Sigma \). Let \( f = k \cdot f_\infty \in L^1(G) \ast L^\infty(G) = C(G) \). Then

\[
\|f\|_\infty \leq \|k\|_1 \|f_\infty\|_\infty \leq 36\|g\|_2 \leq 36(\|h\|_2 + \delta) = 37\|A\|_2.
\]

Since \( k \) is central in \( L^1(G) \), \( \hat{k}(\sigma) \) is seen to be a scalar multiple of the identity in \( \mathcal{B}(\mathcal{K}_\sigma) \). Write \( \hat{k}(\sigma) = c_\sigma \mathbf{I}_\sigma \). Then

\[
\|\hat{f}(\sigma)\|_2 = |c_\sigma| \|\hat{f}_\infty(\sigma)\|_2 \geq |c_\sigma| \|\hat{g}(\sigma)\|_2 = \|A^\sigma\|_2.
\]

A corollary of our theorem is a generalization of Carleman's theorem for the circle [1]. (See also [2, 37.22(k)] for another proof of this corollary.) For the statement of this corollary, some additional notation is necessary. Let \( A \in \mathcal{B}(\mathcal{K}_\sigma) \) and let \( |A| \) be the (unique) positive-definite square root of \( AA^* \). Let \( \lambda_1, \ldots, \lambda_d \) be the eigenvalues of \( |A| \). Define the von Neumann norms on \( \mathcal{B}(\mathcal{K}_\sigma) \) by

\[
\|A\|_{\phi, r} = \left( \sum_i (\lambda_i)^r \right)^{1/r} \quad (1 \leq r < \infty).
\]

(Note that \( \|\|_{\phi, r} \) is the same as \( \|\|_{2, \phi} \).

For \( A = (A^\sigma) \in \mathcal{E}(\Sigma) \), define

\[
\|A\|_p = \left( \sum_\sigma d_\sigma \|A^\sigma\|_{\phi, r}^p \right)^{1/p} \quad (1 \leq r < \infty).
\]

Let \( \mathcal{E}^p(\Sigma) = \{ A \in \mathcal{E}(\Sigma) : \|A\|_p < \infty \} \).

**Corollary.** There is a continuous function \( f \) defined on \( G \) for which \( \hat{f} \notin \mathcal{E}^{2,p}(\Sigma) \) for \( 1 \leq p < 2 \).

**Proof.** Let \( \{\sigma_1, \sigma_2, \sigma_3, \ldots\} \) be a countably infinite subset of \( \Sigma \), with no repetitions. Let \( A_n^\sigma \) be the \( d_{\sigma_1} \times d_{\sigma_2} \)-matrix whose (1, 1)-entry is \( 1/\sqrt{n d_{\sigma_2}} \log n \) and whose other entries are zero. Let \( A^\sigma = 0 \) for all other \( \sigma \in \Sigma \). Then \( A \in \mathcal{E}^{2,2}(\Sigma) \).

By the preceding theorem, there exists a function \( f \in C(G) \) with

\[
\|\hat{f}(\sigma)\|_2 \geq \|A^\sigma\|_2 \quad \text{for every } \sigma \in \Sigma.
\]

For every \( \sigma \in \Sigma \),

\[
\|\hat{f}(\sigma)\|_{\phi, r} \geq \|\hat{f}(\sigma)\|_2 \geq \|A^\sigma\|_2 = \|A^\sigma\|_{\phi, r}.
\]

Since

\[
d_{\sigma_n} \|A^\sigma\|_{\phi, r} = d_{\sigma_n} \left( \frac{1}{\sqrt{n d_{\sigma_n}} \log n} \right)^r \geq \frac{1}{n \log n}
\]

for all sufficiently large \( n \), \( \hat{f} \notin \mathcal{E}^{2,p}(\Sigma) \).

**Acknowledgements.** An earlier version of this paper was written while the author was a graduate student at Kansas State University. I am grateful to the mathematics department there for their support and encouragement. Special thanks are due Karl Stromberg for his guidance. I would also like to thank Edwin Hewitt for his valuable suggestions.
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