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ANOTHER INTERESTING PROPERTY CONCERNING THE PROBABILITY MEASURES ON THE RATIONALS

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ABSTRACT. Let X be a perfect, complete, separable metric space and $P(X)$ denote the space of Borel probability measures on X equipped with the topology of weak convergence. If Y is a countable dense subset of X then $P(Y)$ is not a $G_{\delta\sigma}$ subset of $P(X)$. Furthermore if X is separable, complete and metric, and $Y \subseteq X$, and $P(Y)$ is a $G_{\delta\sigma}$ subset of $P(X)$, then $P(Y)$ is in fact a G_δ subset of $P(X)$.

I. Introduction. Let X be a separable metric space and $P(X)$ denote the space of Borel probability measures on X. The topology on $P(X)$ is the topology of weak convergence, i.e. $\lim m_i = m$ iff $\limsup m_i(F) \leq m(F)$ for all closed $F \subseteq X$. Equivalent versions of the topology and other properties of $P(X)$ can be found in [**4**, pp. 39–50].

In [**5**] Preiss showed that if Y is taken to be the rationals (or more generally is first category in itself) then Y is not Prohorov. Preiss then used this result to show that if Y is any separable metric coanalytic space, then Y is Prohorov implies that Y is topologically complete (which is equivalent to $P(Y)$ being topologically complete).

In this paper it is shown that if Y is a countable dense subset of a perfect Polish space X, then $P(Y)$ is not a $G_{\delta\sigma}$ subset of $P(X)$. Then, using Theorem 2 of [**5**], it is shown that if $Y \subseteq X$ (Polish) and $P(Y)$ is a $G_{\delta\sigma}$ subset of $P(X)$, then $P(Y)$ is in fact a G_δ subset of $P(X)$.

This result is of the same flavor as that of Luther in [**3**] in which he showed that local compactness of $P(Y)$ (actually the existence of a compact neighborhood) implies that $P(Y)$ is compact. It is not difficult using Luther's result to show if $P(Y)$ is σ-compact then $P(Y)$ is compact.

These three results can be summarized as follows: if Y is separable and metric then:

(1) if Y is CA and Prohorov then $P(Y)$ is absolute G_δ,

(2) if $P(Y)$ is absolute $G_{\delta\sigma}$ then $P(Y)$ is absolute G_δ,

(3) if $P(Y)$ is absolute F_σ then $P(Y)$ is absolute closed.

The necessary facts concerning absolute Borel class can be found in [**2**, p. 339].

II. Background. Two background theorems require explicit mention. The first is due to Prohorov and may be found in [**4**, p. 46, Theorem 6.5].

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THEOREM A (PROHOROV). *If X is separable and metric, then $P(X)$ is topologically complete iff X is.*

The second may be found in [**6**, p. 430].

THEOREM B. *If X is complete, then $F \subseteq X$ is topologically complete iff F is a G_δ subset of X.*

In [7] Varadarajan notes that a basis for the topology of weak convergence is formed by finite intersections of $\{m \in P(X)\colon m(U) > m_0(U) - \varepsilon\}$ where m_0 is some fixed probability measure, U is open in X and $\varepsilon > 0$. For our purposes we need to show that sets of a more restrictive form constitute a basis for $P(X)$.

LEMMA 1. *Sets of the form $N^*(m_0; \varepsilon; U_1, \ldots, U_n) = \{m \in P(X)\colon m(U_i) > m_0(U_i) - \varepsilon,\ i = 1, n\}$ where $\varepsilon > 0$ and $U_1, \ldots, U_n$ are disjoint open subsets of X such that $\sum_{i=1}^n m_0(U_i) = 1$, form a basis for the weak topology on $P(X)$.*

PROOF. The usual basis for the weak topology on $P(X)$ is defined by sets of the form $N(m_0; \varepsilon; f_1, \ldots, f_n) = \{m \in P(X)\colon |\int f_j\, dm - \int f_j\, dm_0| < \varepsilon,\ j = 1, n\}$ where the functions f_j are required to be bounded continuous functions from X into the reals (see [**4**]). For convenience assume that the f_j are into $[0, 1]$.

As $N(m_0; \varepsilon; f_1, \ldots, f_n) = \bigcap_{i=1}^n N(m_0; \varepsilon; f_i)$, we need only consider $N(m_0; \varepsilon; f_i)$ and show that intersections may be accommodated.

Consider $N(m_0; \varepsilon; f)$ where f is taken to be from X into $[0, 1]$. Pick an integer $n > 2$ so that $(1/n) < (\varepsilon/8)$. Pick numbers $0 < a_0 < a_1 < \cdots < a_{n+1} = 1$ so that the mesh of the subdivision is less than $(1/n)$ and $m_0([f \in \{a_0, \ldots, a_n\}]) = 0$. Note that the last property can be obtained since $n + 1$ points are chosen and only countably many points are the image under f of sets of positive m_0-measure.

Let $U_0 = [f \in [0, a_0)]$, $U_i = [f \in (a_{i-1}, a_i)]$ for $1 \leq i \leq n$ and $U_{n+1} = [f \in (a_n, 1]]$. Note $\sum_{i=0}^{n+1} m_0(U_i) = 1$. Pick $0 < \delta < 1$ so that $\delta(n + 1) < (\varepsilon/8)$.

Consider $m \in N^*(m_0; (\delta/2^{n+1}); U_0, \ldots, U_{n+1})$. If $m(U_{n+1}) \geq m_0(U_{n+1}) + \delta$ then

$$\sum_{i=0}^{n+1} m(U_i) \geq \sum_{i=0}^{n} = \left[m_0(U_i) - \delta/2^{n+1}\right] + m_0(U_{n+1}) + \delta$$
$$= 1 - \left[(n + 1)/2^{n+1}\right]\delta + \delta > 1.$$

Clearly this holds for each U_i, so $|m(U_i) - m_0(U_i)| < \delta$ for $0 \leq i \leq n + 1$. This gives the inequalities

$$\sum_{i=1}^{n+1} m_0(U_i) a_{i-1} \leq \int f\, dm_0 \leq \sum_{i=0}^{n+1} m_0(U_i) a_i$$

and

$$\sum_{i=1}^{n+1} \left[m_0(U_i) - \delta\right] a_{i-1} \leq \int f\, dm \leq \sum_{i=0}^{n+1} \left[m_0(U_i) + \delta\right] a_i.$$

Therefore

$$[m_0(U_0) + \delta](-a_0) - \sum_{i=1}^{n+1} [m_0(U_i)(a_i - a_{i-1}) + \delta a_i] \leqslant \int f\,dm_0 - \int f\,dm$$

$$\leqslant m_0(U_0)a_0 + \sum_{i=1}^{n+1} [m_0(U_i)(a_i - a_{i-1}) + \delta a_{i-1}].$$

Now $a_0 < (\varepsilon/8)$ implies $m(U_0)a_0 < (\varepsilon/8)$ and $\delta a_0 < (\varepsilon/8)$. Furthermore,

$$\sum_{i=1}^{n+1} m_0(U_i)(a_i - a_{i-1}) \leqslant \frac{1}{n} \sum_{i=1}^{n+1} m(U_i) \leqslant \frac{1}{n} < \frac{\varepsilon}{8}$$

and

$$\sum_{i=1}^{n+1} \delta a_{i-1} < \sum_{i=1}^{n+1} \delta a_i = \delta \sum_{i=1}^{n+1} a_i < \delta(n+1) < \frac{\varepsilon}{8}.$$

So $|\int f\,dm_0 - \int f\,dm| < \varepsilon$.

In order to accommodate

$$N(m_0; \varepsilon; f_i) \cap N(m_0; \varepsilon; f_j)$$

consider $N^*(m_0; \delta_1; U_1, \ldots, U_n)$ and $N^*(m_0; \delta_2; O_1, \ldots, O_k)$. Let $U_{ij} = U_i \cap O_j$ and $\delta = (\min\{\delta_1, \delta_2\}/\max\{k, n\})$. Note that for $m \in P(X)$, $m(U_i) \geqslant \sum_{j=1}^{k} m(U_{ij})$ and $m(O_j) \geqslant \sum_{i=1}^{n} m(U_{ij})$ and for m_0 these are equalities.

Suppose $m \in N^*(m_0; \delta; U_{11}, \ldots, U_{nk})$. So $m(U_{ij}) > m_0(U_{ij}) - \delta$ implies

$$m(U_i) \geqslant \sum_{j=1}^{k} m(U_{ij}) > \sum_{j=1}^{k} [m_0(U_{ij}) - \delta]$$

$$= m_0(U_i) - k\delta \geqslant m_0(U_i) - \delta_1.$$

Similarly $m(O_j) > m_0(O_j) - \delta_2$, so m belongs to the intersection of $N^*(m_0; \delta_1; U_1, \ldots, U_n)$ and $N^*(M_0; \delta_2; O_1, \ldots, O_k)$.

$N^*(m_0; \varepsilon; U_1, \ldots, U_n)$ is open since it is equal to

$$\bigcap_{i=1}^{n} \{m \in P(X): m(U_i) > m_0(U_i) - \varepsilon\}$$

and these sets are open. □

The following lemma follows directly from the Baire Category Theorem. It is stated as a lemma in the form that will be used in the example to follow.

LEMMA 2. *Suppose X is topologically complete, V is a closed subset of X, $O \subseteq V$ open relative to V, $N \subseteq V$ that is first category in V and $\langle K_i \rangle$ is a sequence of closed sets in X such that $O - N \subseteq \bigcup_{i=1}^{\infty} K_i$. There exists a set $U \subseteq O$ with the following properties*:

(i) *U is open relative to O,*

(ii) *$\bar{U} \subseteq O$ ($\bar{U}$ denoting the closure of U in X),*

(iii) *$\bar{U} \subseteq K_j$ for some j,*

(iv) *the diameter of $\bar{U} < (1/2^i)$ where i is an arbitrary natural number.*

III. Main result. We now introduce the following notation. Let X be a perfect, complete, separable metric space and Y a countable dense subset of X. Let $Y = \{y_1, y_2, \ldots\}$ and $Y_i = \{y_1, y_2, \ldots, y_i\}$. Let $i_1, i_2, i_3, \ldots, i_n$ denote an increasing sequence of positive integers, $C_{ij} = \{m \in P(X) \mid m(Y_i) \geqslant 1 - (1/j)\}$, $V_n = \bigcap_{k=1}^{n} C_{i_k k}$ and $C_j = \bigcup_{i=1}^{\infty} C_{ij}$. Note that C_{ij} is closed in $P(X)$ and $\bigcap_{j=1}^{\infty} C_j = P(Y)$. Assume that $P(X)$ has been equipped with a complete metric.

LEMMA 3. *$V_n \cap C_{in+1}$ is nowhere dense in V_n.*

PROOF. $V_n \cap C_{in+1}$ is closed in V_n so it remains to show that if $m_0 \in V_n \cap C_{in+1}$ then any neighborhood of m_0 contains a point of $V_n - C_{in+1}$.

Consider $N^*(m_0; \varepsilon; U_1, \ldots, U_k)$. Since X is perfect and complete and Y is countable we may choose a_j such that $a_j \in U_j - Y$ for $1 \leqslant j \leqslant k$.

Case 1. $m_0(Y_{i_n}) < 1 - 1/(n+1)$. Define $m \in P(X)$ by the following: $m(\{y_j\}) = m_0(\{y_j\})$ for $y_j \in Y_{i_n}$ and $m(\{a_j\}) = m_0(U_j) - m(U_j \cap Y_{i_n})$ for $1 \leqslant j \leqslant k$. Clearly $m(U_j) = m_0(U_j)$ and m agrees with m_0 on Y_{i_n} but $m(Y) < 1 - 1/(n+1)$. So $m \in N^*(m_0; \varepsilon; U_1, \ldots, U_k) \cap [V_n - C_{in+1}]$.

Case 2. $m_0(Y_{i_n}) \geqslant 1 - (1/n+1)$. Let t denote the first integer between 1 and i_n inclusive with the property that $\sum_{j=1}^{t} m_0(\{y_j\}) \geqslant 1 - 1/(n+1)$. Pick $0 < r < 1$ so that

$$1 - \frac{1}{n} < \sum_{j=1}^{t-1} m_0(\{y_j\}) + r m_0(\{y_t\}) < 1 - \frac{1}{n+1}.$$

Define m by the following: $m(\{y_j\}) = m_0(\{y_j\})$ for $j < t$, $m(\{y_t\}) = r m_0(\{y_t\})$ and $m(\{a_j\}) = m_0(U_j) - m(U_j \cap Y_t)$ for $1 \leqslant j \leqslant k$. Now, $m(U_j) = m_0(U_j)$ and m agrees with m_0 on Y_{t-1}. Since $m_0 \in V_n$, $m(Y_{i_j}) \geqslant 1 - (1/j)$ for $i_j < t$. If $t \leqslant i_j \leqslant i_n$ then $m(Y_{i_j}) \geqslant 1 - (1/n) \geqslant 1 - (1/j)$. Hence, $m \in V_n$. On the other hand, $m(Y) < 1 - 1/(n+1)$ implies that $m \notin C_{in+1}$. This completes the proof. □

LEMMA 4. *$V_n \cap C_{n+1}$ is dense in V_n.*

PROOF. Let $m_0 \in V_n$ and consider $N^*(M_0; \varepsilon; U_1, U_2, \ldots, U_k)$. Since Y is dense and X is perfect, $(Y - Y_{i_n}) \cap U_j \neq 0$. Pick $i > i_n$ with the property that $(Y_i - Y_{i_n}) \cap U_j \neq 0$, $1 \leqslant j \leqslant k$. Pick $a_j \in (Y_i - Y_{i_n}) \cap U_j$. Define m by the following: $m(\{y_j\}) = m_0(\{y_j\})$ for $1 \leqslant j \leqslant i_n$ and $m(\{a_j\}) = m_0(U_j) - m(Y_{i_n} \cap U_j)$. Clearly $m(Y_i) = 1$, so $m \in C_{n+1}$. Furthermore m agrees with m_0 on Y_{i_n} implies $m \in V_n$ and $m(U_j) = m_0(U_j)$ implies $m \in N^*(m_0; \varepsilon; U_1, \ldots, U_k)$. □

EXAMPLE 5. In this example the notation introduced before Lemma 3 will be used. It will be shown that $P(Y)$ is not a $G_{\delta\sigma}$ subset of $P(X)$. Assume that $P(X)$ has been equipped with complete metric.

Suppose $P(Y)$ is a $G_{\delta\sigma}$ subset of $P(X)$. Then $P(X) - P(Y)$ is an $F_{\sigma\delta}$ subset of $P(X)$ and there exist closed sets $\langle K_{ij} \rangle$ such that $P(Y)^c = \bigcap_{j=1}^{\infty} K_j$ where $K_j = \bigcup_{i=1}^{\infty} K_{ij}$.

Using Lemmas 2, 3 and 4 a sequence of sets, $\langle \bar{O}_n \rangle$, will be constructed that have the property that their intersection belongs to both $P(Y)$ and $P(Y)^c$. Recall the notation introduced before Lemma 3.

Let $i_1 = 1$. Then $V_1 = C_{i_1 1}$ is a closed subset of $P(X)$. By Lemma 3 we have that $V_1 \cap C_2$ is a first category F_σ set with respect to V_1. Let $O_1 = P(X)$. Now $O_1 - (V_1 \cap C_2) \subseteq C_2^c \subseteq P(Y)^c \subseteq K_1$ (note $O_1 \subseteq V_1$ and in general O_n will be a subset of V_n). By Lemma 2 there exists a set $U_1 \subseteq O_1$ having properties (i)–(iv) as stated in the lemma. For property (iv) require that the diameter of $\overline{U}_1 < \frac{1}{2}$. Note that K_1 corresponds to the union of the K_i of Lemma 2.

Since U_1 is open in O_1 and, by Lemma 4, C_2 is dense in V_1, we can pick $i_2 > i_1$ so that $C_{i_2 2}$ intersects U_1. Let $V_2 = V_1 \cap C_{i_2 2}$ and $O_2 = U_1 \cap V_2$. Note that O_2 is open in V_2 and V_2 is a closed subset of $P(X)$ which is complete. Proceed to the $n + 1$ step.

Assume that $i_1 < \cdots < i_n$ have been defined, $V_n = \bigcap_{k=1}^n C_{i_k k}$, $O_1 \supseteq O_2 \cdots O_n$ have been defined so that the diameter of $O_k < (1/2^{k-1})$ and $O_k \subseteq V_k$ is open relative to V_k. By Lemmas 3 and 4, $V_n \cap C_{n+1}$ is a first category F_σ subset of V_n and is dense in V_n. $O_n - (V_n \cap C_{n+1}) \subseteq C_{n+1}^c \subseteq P(Y)^c \subseteq K_n$, so by Lemma 2 there exists $U_n \subseteq O_n$ having properties (i)–(iv) (where the diameter of $\overline{U}_n < (1/2^n)$). Pick $i_{n+1} > i_n$ so that $C_{i_{n+1} n+1}$ intersects U_n. Let $V_{n+1} = V_n \cap C_{i_{n+1} n+1}$ and $O_{n+1} = U_n \cap V_{n+1}$.

The sets, $\overline{O}_1 \supseteq \overline{O}_2 \supseteq \cdots$, are nonempty and the diameter of $\overline{O}_n < (1/2^{n-1})$ for $n > 1$. Since $P(X)$ is complete there exists $m \in \bigcap_{n=1}^\infty \overline{O}_n$. For each n, $O_n \subseteq V_n$ and V_n is closed so $\overline{O}_n \subseteq V_n$ and $m \in V_n$. Therefore, for each n, $m \in C_{i_n n}$, so $m \in C_n$ and $m \in \bigcap_{n=1}^\infty C_n = P(Y)$.

On the other hand $\overline{O}_n \subseteq \overline{U}_{n-1}$ for $n \geqslant 2$. By property (iii) of Lemma 2, $\overline{U}_{n-1} \subseteq K_{n-1}$. This gives that $m \in K_n$ for each n and $m \in \bigcap_{n=1}^\infty K_n = P(Y)^c$. □

THEOREM 6. *If X is separable metric and coanalytic but not topologically complete, then X contains a countable, dense in itself, G_δ subspace.*

PROOF. See [**1** or **5**].

THEOREM 7. *If X is a complete separable metric space and $Y \subseteq X$ and $P(Y)$ is a $G_{\delta\sigma}$ subset of $P(X)$ then $P(Y)$ is in fact a G_δ subset of $P(X)$.*

PROOF. Suppose Y fails to be a G_δ subset of X (and hence, by Theorem B, Y is not topologically complete). Y and X are homeomorphic to the degenerate measures (point masses) in $P(Y)$ and $P(X)$, see [**4**], so Y is a $G_{\delta\sigma}$ subset of X. Since X is complete, Y is coanalytic. By Theorem 6 there exists a G_δ subset of X, call it G, such that $G \cap Y$ is countable and dense in itself. Let $Y_1 = G \cap Y$ and $X_1 = \overline{Y}_1 \cap G$, i.e. X_1 is the closure of Y_1 in G. X_1 is dense in itself and topologically complete. Y_1 is a countable dense subset of X_1.

Since $Y_1 = Y \cap X_1$ we have that $P(Y_1) = P(Y) \cap P(X_1)$. By hypothesis $P(Y)$ is a $G_{\delta\sigma}$ in $P(X)$, so $P(Y_1)$ is a $G_{\delta\sigma}$ in $P(X_1)$. On the other hand, X_1 and Y_1 have the properties specified in Example 5 in which it was shown that $P(Y_1)$ is not a $G_{\delta\sigma}$ in $P(X_1)$; hence, Y must be a G_δ subset of X and therefore, by Theorems A and B, $P(Y)$ is a G_δ subset of $P(X)$. □

REMARK. It is not difficult to show that if Y belongs to the αth multiplicative Borel class with respect to X, then $P(Y)$ belongs to the αth multiplicative Borel class with respect to $P(X)$. In light of statements (2) and (3) of §I, is the following true?

If $P(Y)$ belongs to the βth, $\beta > 0$, additive Borel class with respect to $P(X)$, then there exists $\alpha < \beta$ such that $P(Y)$ belongs to the αth multiplicative Borel class with respect to $P(Y)$.

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