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# COMPACT DISPERSED SPACES AND THE $\alpha$-LEFT PROPERTY[1]

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ABSTRACT. The concept of an $\alpha$-left space was introduced by Arhangel'skij in [$A_1$], where, among other results, it was shown that every $T_2$ compact $\alpha$-left space is dispersed. In [$A_2$] an example was given to show that, assuming the continuum hypothesis CH, not every $T_2$ compact dispersed space is $\alpha$-left. The aim of this article is to obtain some sufficient conditions for $u$ $T_2$ space to be $\alpha$-left and to construct a large class (which contains all products of uncountable ordinals) of compact $T_2$ dispersed spaces which are not $\alpha$-left.

**1. Introduction, notation and preliminary results.** A space $X$ is said to be left-separated (resp. almost left-separated) if there exists a well-order $\{x_\alpha: \alpha < \lambda\}$ ($\lambda$ ordinal) of $X$ such that every initial segment (resp. every closed initial segment) is closed in $X$. A well-order $<$ which makes $X$ left-separated (resp. almost left-separated) is called a left (resp. almost left) well-order. In order that our terminology coincides with that of [$A_1$] we will abbreviate almost left-separated to $\alpha$-left. Clearly every left-separated space is $\alpha$-left, but the converse is false: any ordinal is an $\alpha$-left space with its natural order, but Gerlits and Juhász [GJu] have shown that a compact $T_2$ left-separated space is both scattered and sequential, and hence the only left-separated ordinals are countable.

If $\tau$ is an ordinal, $\tau + 1$ will denote the ordinal successor of $\tau$ while if $\tau$ is a cardinal $\tau^+$ will denote its cardinal successor.

When considered as topological spaces, all ordinals are assumed to have the order topology. *All spaces are assumed to be Hausdorff.* If $A$ and $B$ are ordered sets the lexicographic ordering of $A \times B$ is always assumed to be by first differences, i.e. $(a, b) < (c, d)$ if $a < c$ or if $a = c$ and $b < d$. The proofs of the following lemmas are easy and are left to the reader.

1.1 LEMMA. *If $\{X_\lambda: \lambda \in \Lambda\}$ is a family of $\alpha$-left (respectively left-separated) spaces then the disjoint topological union of the $X_\lambda$'s is also $\alpha$-left (resp. left-separated).*

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1.2 LEMMA. *If a space $X$ is the union of a countable family of mutually disjoint closed $\alpha$-left (resp. left-separated) subspaces then $X$ is $\alpha$-left (resp. left-separated).*

1.3 LEMMA. *If $X$ is an $\alpha$-left (resp. left-separated) space and $p \in X$, then there is an almost left well-order (resp. left well-order) of $X$ in which $p$ is the first point.*

**2. A class of dispersed compact $T_2$ spaces which are not $\alpha$-left.** Recall that the tightness of a ($T_1$) space $X$ is the least cardinal $\tau$ for which the following holds: For every $Y \subset X$ and each $p \in \mathrm{cl}_X Y$, there is a subset $A$ of $Y$ such that $|A| \leq \tau$ and $p \in \mathrm{cl}_X A$.

Our aim in this section is to prove the following result.

2.1 THEOREM. *Let $X$ be a space with infinite tightness $\tau$. The product space $X \times \tau^+$ is $\alpha$-left if and only if $X$ is left-separated.*

The proof will consist of a sequence of lemmas, but first we note some corollaries. As mentioned in the introduction, an ordinal is left-separated if and only if it is countable; thus we have

2.2 COROLLARY. *$\omega_1 \times \omega_1$ is not $\alpha$-left.*

Since every subspace of an $\alpha$-left space is $\alpha$-left (the restriction of the almost left well-order to the subspace has the required property) we have

2.3 COROLLARY. *If $\lambda$ and $\mu$ are ordinals with the order topology, then $\lambda \times \mu$ is $\alpha$-left if and only if $\lambda$ or $\mu$ is countable.*

Thus $(\omega_1 + 1) \times (\omega_1 + 1)$ is a dispersed compact $T_2$ space which is not $\alpha$-left, showing that CH is not needed in [**A**$_2$, Corollary 8].

We now begin the proof of the theorem with

2.4 LEMMA. *If $X$ is left-separated and $Y$ is an $\alpha$-left space then $X \times Y$ is $\alpha$-left.*

PROOF. Order $X$ with a left well-order and $Y$ with an almost left well-order. It is easy to verify that the lexicographic order on $X \times Y$ is an almost left well-order.

2.5 LEMMA. *Let $Y$ be a topological space, $\lambda$ an ordinal, $g\colon \lambda \to Y$ an almost left well-order of $Y$. Let $\tau$ be an infinite cardinal and $\varphi\colon \tau^+ \to Y$ a homeomorphism of $\tau^+$ onto $\varphi[\tau^+] = L$. Denote by $\alpha_0$ the least ordinal no greater than $\lambda$ such that $\varphi^{-1}(g[\alpha_0])$ is cofinal in $\tau^+$. Then:*

(i) $\mathrm{cof}(\alpha_0) \geq \tau^+$.

(ii) *$g[\alpha_0] \cap L$ is closed in $L$.*

(iii) *If $\psi\colon \tau^+ \to Y$ is a homeomorphism of $\tau^+$ onto $M = \psi[\tau^+]$ and $M \cap L = \varnothing$, then $\psi^{-1}(g[\alpha_0])$ is not cofinal in $\tau^+$.*

PROOF. (i) This easy proof is left to the reader.

(ii) $g[\alpha_0 + 1] = g[\alpha_0] \cup \{g(\alpha_0)\}$ is closed in $Y$ since $g$ is an almost left well-order. If $g(\alpha_0) \notin L$, then (ii) holds. If $g(\alpha_0) \in L$, then $g(\alpha_0)$ has a neighbourhood $U$ in $L$ of cardinality no greater than $\tau$. By (i) there exists $\beta < \alpha_0$ such that $g(\gamma) \notin U$ for every $\gamma > \beta$. Since $g$ is bijective we have $g(\alpha_0) \in U \cap (Y - g[\beta + 1])$, this being an open subset of $L$ which misses $g[\alpha_0] \cap L$.

(iii) For $\alpha < \alpha_0$ we write $F_\alpha = g[\alpha + 1] \cap L$ and $G_\alpha = g[\alpha + 1] \cap M$; if we assume that $\psi^{-1}(g[\alpha_0])$ is cofinal in $\tau^+$, both $F_\alpha$ and $G_\alpha$ are nonempty for large $\alpha$, say for $\alpha \geqslant \beta_0$; for every such $\alpha$ we put $a_\alpha = \sup \psi^{-1}(F_\alpha)$, and $b_\alpha = \sup \psi^{-1}(G_\alpha)$. Since $\{a_\alpha: \beta_0 \leqslant \alpha < \alpha_0\}$ and $\{b_\alpha: \beta_0 \leqslant \alpha < \alpha_0\}$ are cofinal in $\tau^+$, it is possible to construct recursively a sequence $\beta_0 < \beta_1 < \beta_2 < \cdots$ of ordinals smaller than $\alpha_0$ such that $a_{\beta_0} < a_{\beta_1} < a_{\beta_2} < \cdots$ and $b_{\beta_0} < b_{\beta_1} < b_{\beta_2} < \cdots$. If $\beta = \sup\{\beta_n: n \in \omega\}$ then $\beta < \alpha_0$ (by (i)) and $g(\beta) \notin L$ or $g(\beta) \notin M$ (or both). If $g(\beta) \notin L$ then

$$F_\beta = g[\beta + 1] \cap L = g[\beta] \cap L = \bigcup_{n \in \omega} g[\beta_n + 1] \cap L = \bigcup_{n \in \omega} F_{\beta_n};$$

but $a_\beta = \sup\{a_{\beta_n}: n \in \omega\} \notin \varphi^{-1}(F_\beta)$, which is a contradiction since $\varphi^{-1}(F_\beta)$ is closed in $\tau^+$.

2.6 Lemma. *Let $X$ be a $T_1$ space with infinite tightness $\tau$. If $\tau^+ \times X$ is $\alpha$-left then $X$ is left-separated.*

Proof. Let $\lambda$ be an ordinal and $g: \lambda \to \tau^+ \times X$ an almost left well-order of $\tau^+ \times X$. Given $x \in X$, $R_x = \tau^+ \times \{x\}$ is a closed copy of $\tau^+$ in $\tau^+ \times X$: let $\alpha_x$ be the smallest ordinal no greater than $\lambda$ such that $g[\alpha_x]$ is cofinal in $R_x$. By Lemma 2.5, $x \to \alpha_x$ is an injection of $X$ into $\lambda + 1$, and for every $x \in X$ the set $C_x = g[\alpha_x] \cap R_x$ is closed and cofinal in $R_x$. Well-order $X$ by defining $x \lhd y$ if and only if $\alpha_x < \alpha_y$. We claim that $\lhd$ is a left well-order of $X$. To prove this, let $a \in X$ and put $A = \{x \in X: x \lhd a\}$. We wish to prove that if $p \in \operatorname{cl}_X(A)$ then $p \in A$. Since $X$ has tightness $\tau$, there exists a subset $B = \{a(\xi): \xi \in \tau\}$ of $A$ such that $p \in \operatorname{cl}_X(B)$; excluding trivial cases, we can assume that $B$ is infinite and that $a(\xi) \neq p$ for each $\xi \in \tau$. Put $G = \pi_1(C_p) \cap (\bigcap_{\xi \in \tau} \pi_1(C_{a(\xi)}))$, where $\pi_1: \tau^+ \times X \to \tau^+$ denotes the first projection. $G$ is the intersection of $\tau$ closed cofinal subsets of $\tau^+$ and hence by [**J**, Lemma 7.4], it is closed and cofinal in $\tau^+$. Then $G \times \{p\} \subset \operatorname{cl}(\bigcup_{\xi \in \tau}(G \times \{a(\xi)\}))$, and $G \times \{a(\xi)\} \subset g[\alpha_{a(\xi)}]$ for each $\xi \in \tau$. Let $\delta = \sup\{\alpha_{a(\xi)}: \xi \in \tau\}$. Then either $\delta < \lambda$ or $\delta = \lambda$. If $\delta < \lambda$, then $g[\delta + 1]$ is a closed subset of $\tau^+ \times X$ which contains $\bigcup_{\xi \in \tau}(G \times \{a(\xi)\})$, and hence also contains $G \times \{p\}$, which is cofinal in $R_p$; thus $\delta \geqslant \alpha_p$. If $\delta = \alpha_{a(\xi)}$ for some $\xi \in \tau$, then $\alpha_p \leqslant \alpha_{a(\xi)}$ and hence $p \trianglelefteq a(\xi) \in A$ and so $p \in A$; if not, then $\operatorname{cof} \delta \leqslant \tau$, and hence $\delta > \alpha_p$ (since by Lemma 2.5(i), we have $\operatorname{cof} \alpha_p \geqslant \tau^+$). Thus $\alpha_p < \alpha_{a(\xi)}$ for some $\xi \in \tau$, and this implies that $p \lhd a(\xi) \in A$, and so $p \in A$.

If $\delta = \lambda$, the above argument is valid if $\delta = \lambda = \alpha_{a(\xi)}$ for some $\xi \in \tau$. If $\delta \neq \alpha_{a(\xi)}$ for each $\xi \in \tau$, then $\delta = \lambda = \alpha_p$; but then $\operatorname{cof} \alpha_p = \operatorname{cof} \delta \leqslant \tau$, which is impossible by Lemma 2.5(i).

2.7 Proof of theorem 2.1. Sufficiency (with no restriction on tightness) follows from 2.4; necessity from 2.6.

**3. $\alpha$-left ordered spaces.** As was pointed out in [**A$_1$**], not every ordered space is $\alpha$-left; indeed, not even every zero-dimensional ordered space is $\alpha$-left, although every $\alpha$-left ordered space is certainly zero dimensional. Our aim in this section is to prove that every dispersed ordered space is $\alpha$-left.

Recall that every dispersed space $X$ can be written as a disjoint union of the form $X = \bigcup \{D_\xi: \xi < \rho\}$ (where $\rho$ is an ordinal) and where for each $\xi < \rho$, $D_\xi$ is the set of isolated points of the closed subspace $X_\xi = \bigcup \{D_\alpha: \alpha \geqslant \xi\}$. The least ordinal $\rho$ such that $D_\rho = \varnothing$ is called the dispersal order of $X$. It is known (see [**R**]), that if $X$ is compact then $\rho = \lambda + 1$ and $D_\lambda$ is finite.

3.1 THEOREM. *Every compact dispersed ordered space is $\alpha$-left.*

PROOF. First note that every totally disconnected ordered space has a base of open and closed intervals which, if the space is compact, must have first and last elements.

The proof is by induction on $\lambda = \rho - 1$, where $\rho$ is the dispersal order of $X$ (see above). If $\lambda = 0$, then $D_0 = X$ which is thus a finite Hausdorff space and hence is $\alpha$-left. We now assume that every compact dispersed ordered space of dispersal order $\delta - 1 = \mu < \lambda$ is $\alpha$-left. Since $D_\lambda = \{p_1, \ldots, p_m\}$ for some integer $m$, we can split $X$ into a disjoint union of $m$ open and closed intervals, each containing precisely one $p_k$. We are thus reduced to the case $D_\lambda = \{p\}$, a singleton. In this case $X = [a, p] \cup [p, b]$ where $a = \min X$ and $b = \max X$; we will first show that $[a, p]$ is $\alpha$-left. Let $\{U_\alpha: \alpha < \kappa\}$ be a nested neighbourhood basis for $p$ in $[a, p]$, consisting of open and closed intervals $[a_\alpha, p]$, indexed by a (regular) cardinal $\kappa$. For each $\alpha \in \kappa$, let $Y_\alpha = X - U_\alpha = [a, b_\alpha]$ (where $b_\alpha$ is the immediate predecessor of $a_\alpha$ in the order of $X$). By the inductive hypothesis, for each $\alpha \in \kappa$ there exists an ordinal $\gamma_\alpha$ which almost left separates $Y_\alpha$. We will show that this can be done in such a way that $\alpha < \beta < \kappa$ implies that $Y_\alpha$ is an initial segment of $Y_\beta$; this being possible, $(\bigcup_{\alpha \in \kappa} \gamma_\alpha) + 1$ will almost left-separate $[a, p]$. Assume that this has been done for all $\beta < \alpha < \kappa$. If $\alpha$ is a nonlimit ordinal, $\alpha = \zeta + 1$, then $\{Y_\zeta, Y_\alpha - U_\zeta\}$ is a partition of $Y_\alpha$ into open and closed sets, and according to the inductive hypothesis, $Y_\alpha - U_\zeta$ is almost left-separated by some ordinal $\eta_\alpha$. We can then almost left-separate $Y_\alpha$ by $\gamma_\zeta + \eta_\alpha$. If $\alpha$ is a limit ordinal, $\alpha = \bigcup_{\zeta \in \alpha} \zeta$, then we first well-order $Z_\alpha = \bigcup_{\zeta \in \alpha} Y_\zeta$ by $\gamma = \bigcup_{\zeta \in \alpha} \gamma_\zeta$; note that $Y_\alpha \supset Z_\alpha$ and that equality is impossible since $Y_\alpha$ is compact. Then $Y_\alpha - Z_\alpha$ is a nonempty compact ordered dispersed space which is again almost left-separated by some ordinal $\eta_\alpha$, according to the inductive hypothesis. By Lemma 1.3, the well-ordering of $Y_\alpha - Z_\alpha$ may be chosen in such a way that $\min(Y_\alpha - Z_\alpha)$ is the first element. It is easy to check that $\gamma_\alpha = \gamma + \eta_\alpha$ gives the desired almost left-separation of $Y_\alpha$.

In the same way we can prove that $[p, b]$ is $\alpha$-left and so it is clear that $X = [a, p) \cup \{p\} \cup (p, b]$ is $\alpha$-left; well-order $X$ by placing $p$ first, then well-order $[a, p) \cup (p, b]$ by the ordinal sum.

3.2. Purisch [**P**] has proved that every dispersed suborderable space is orderable and has an ordered dispersed compactification. Thus, recalling that the $\alpha$-left property is hereditary, we have

COROLLARY. *Every dispersed suborderable space is $\alpha$-left.*

3.3. The converse result is false however: The rationals are an example of an $\alpha$-left (even left-separated) ordered space which is not dispersed. Using [$\mathbf{A}_1$, Theorem 1] and the fact that the density and the hereditary density of an ordered space are equal, it is easy to show that every ordered $\alpha$-left space $X$ is strongly scattered (i.e. if $Y \subset X$ then $|Y| = |\mathrm{cl}_X(Y)|$). This fact together with Theorem 3.1 and Theorem 3 of [**A**$_1$], imply

COROLLARY. *For a compact ordered space X the following are equivalent*:
(i) *X is $\alpha$-left,*
(ii) *X is dispersed,*
(iii) *X is strongly scattered.*

Observe that $(\omega_1 + 1) \times (\omega_1 + 1)$ is a strongly scattered compact space which is not $\alpha$-left (by 2.2). On the other hand there exist strongly scattered compact $\alpha$-left spaces which are not orderable: $(\omega + 1) \times (\omega_1 + 1)$ is such a space. (It is not orderable since it is not hereditarily normal; it is $\alpha$-left by Lemma 2.1.)

*Problem*. Is every strongly scattered ordered space $\alpha$-left?

3.4. Telgársky [**T**, Theorem 9] has shown that every dispersed metric space is suborderable, hence 3.2 implies that every such space is $\alpha$-left. However (as the referee has pointed out to us), dispersed metric spaces are $\sigma$-discrete; that is, they are representable as the countable union of (mutually disjoint) closed discrete subspaces [**T**, Theorem 8 or **S**, Theorem 11].

Thus Lemma 1.2 implies

PROPOSITION. *Every dispersed metric space is left-separated.*

3.5. It follows from Corollary 2.3 that since no topological product of two uncountable ordinals is $\alpha$-left, no such space is suborderable. This still leaves open the question of whether a product of an uncountable ordinal and a countable ordinal can be suborderable. (The fact that the product of two countable ordinals is always orderable follows from the above-mentioned results of Purisch and Telgársky.) The following theorem answers this question.

3.6 THEOREM. *Let $\lambda$, $\mu$ be ordinals, $\mu \geqslant \omega_1$. Then $\lambda \times \mu$ is suborderable if and only if $\lambda \leqslant \omega$. In this case $\lambda \times \mu$ is actually orderable.*

PROOF. If $\lambda \leqslant \omega$ then $\lambda$ is discrete and the lexicographic order topology of $\lambda \times \mu$, with the modification that every even column has its order reversed, is the product topology. (The modification is only needed if $\mu$ has no last element.)

Suborderable spaces being hereditarily normal, to complete the proof it suffices to show that $X = (\omega + 1) \times \omega_1$ has a nonnormal subspace. Let $Y$ be the subspace of $X$ obtained by deleting from $X$ all points of the form $(\omega, \lambda)$ where $\lambda \in \omega_1$ is a limit ordinal: We claim that $Y$ is nonnormal. (This was pointed out to us by the referee, thus simplifying our original proof of the theorem.) Let $A = \omega \times \Lambda$, $B = \{\omega\} \times N$, where $\Lambda$, $N$ denote the sets of limit and nonlimit ordinals of $\omega_1$ respectively. Clearly $A$ and $B$ are closed disjoint subsets of $Y$. If $f \in C(Y)$ is identically zero on $A$, then it is easy to show that there exists some $\beta_0 \in \omega_1$ such that $f(n, \beta) = 0$ for every $n \in \omega$ and each $\beta \geqslant \beta_0$; but then $f(\omega, \beta) = 0$ for each $\beta \geqslant \beta_0$ which is nonlimit; that is to say, $f$ vanishes on (infinitely many) points of $B$. Thus $A$ and $B$ are not completely separated in $Y$, and hence $Y$ is not normal.

## REFERENCES

[$A_1$] A. V. Arhangel'skij, *On topologies admitting a weak connection with orders*, Soviet Math. Dokl. **19** (1978), 77–81.

[$A_2$] ______, *On α-expanded spaces*, Soviet Math. Dokl. **19** (1978), 336–340.

[GJu] J. Gerlits and I. Juhász, *On left-separated compact spaces*, Comment. Math. Univ. Carolinae **19** (1978), 53–62.

[J] T. Jech, *Set theory*, Academic Press, New York, 1978.

[P] S. Purisch, *Scattered compactifications and the orderability of scattered spaces*, Topology Appl. **12** (1981), 83–88.

[R] W. Rudin, *Continuous functions on compact spaces without perfect subsets*, Proc. Amer. Math. Soc. **8** (1957), 39–42.

[S] A. H. Stone, *Kernel constructions and Borel sets*, Trans. Amer. Math. Soc. **107** (1963), 58–70.

[T] R. Telgársky, *Total paracompactness and paracompact dispersed spaces*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. **16** (1978), 567–572.

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