COMPACT DISPERSED SPACES AND THE \( \alpha \)-LEFT PROPERTY

G. DE MARCO, A. LE DONNE AND R. G. WILSON

Abstract. The concept of an \( \alpha \)-left space was introduced by Arhangel'skij in \( [A_1] \), where, among other results, it was shown that every \( T_1 \) compact \( \alpha \)-left space is dispersed. In \( [A_2] \) an example was given to show that, assuming the continuum hypothesis \( CH \), not every \( T_2 \) compact dispersed space is \( \alpha \)-left. The aim of this article is to obtain some sufficient conditions for a \( T_2 \) space to be \( \alpha \)-left and to construct a large class (which contains all products of uncountable ordinals) of compact \( T_2 \) dispersed spaces which are not \( \alpha \)-left.

1. Introduction, notation and preliminary results. A space \( X \) is said to be left-separated (resp. almost left-separated) if there exists a well-order \( \{ x_a : a < \lambda \} \) (\( \lambda \) ordinal) of \( X \) such that every initial segment (resp. every closed initial segment) is closed in \( X \). A well-order \( < \) which makes \( X \) left-separated (resp. almost left-separated) is called a left (resp. almost left) well-order. In order that our terminology coincides with that of \( [A_1] \) we will abbreviate almost left-separated to \( \alpha \)-left. Clearly every left-separated space is \( \alpha \)-left, but the converse is false: any ordinal is an \( \alpha \)-left space with its natural order, but Gerlits and Juhász \( [GJu] \) have shown that a compact \( T_2 \) left-separated space is both scattered and sequential, and hence the only left-separated ordinals are countable.

If \( \tau \) is an ordinal, \( \tau + 1 \) will denote the ordinal successor of \( \tau \) while if \( \tau \) is a cardinal \( \tau^+ \) will denote its cardinal successor.

When considered as topological spaces, all ordinals are assumed to have the order topology. All spaces are assumed to be Hausdorff. If \( A \) and \( B \) are ordered sets the lexicographic ordering of \( A \times B \) is always assumed to be by first differences, i.e. \( (a, b) < (c, d) \) if \( a < c \) or if \( a = c \) and \( b < d \). The proofs of the following lemmas are easy and are left to the reader.

1.1 Lemma. If \( \{ X_\lambda : \lambda \in \Lambda \} \) is a family of \( \alpha \)-left (respectively left-separated) spaces then the disjoint topological union of the \( X_\lambda \)'s is also \( \alpha \)-left (resp. left-separated).
1.2 Lemma. If a space $X$ is the union of a countable family of mutually disjoint closed $\alpha$-left (resp. left-separated) subspaces then $X$ is $\alpha$-left (resp. left-separated).

1.3 Lemma. If $X$ is an $\alpha$-left (resp. left-separated) space and $p \in X$, then there is an almost left well-order (resp. left well-order) of $X$ in which $p$ is the first point.

2. A class of dispersed compact $T_2$ spaces which are not $\alpha$-left. Recall that the tightness of a $(T_1)$ space $X$ is the least cardinal $\tau$ for which the following holds: For every $Y \subset X$ and each $p \in \text{cl}_X Y$, there is a subset $A$ of $Y$ such that $|A| \leq \tau$ and $p \in \text{cl}_X A$.

Our aim in this section is to prove the following result.

2.1 Theorem. Let $X$ be a space with infinite tightness $\tau$. The product space $X \times \tau^+$ is $\alpha$-left if and only if $X$ is left-separated.

The proof will consist of a sequence of lemmas, but first we note some corollaries. As mentioned in the introduction, an ordinal is left-separated if and only if it is countable; thus we have

2.2 Corollary. $\omega_1 \times \omega_1$ is not $\alpha$-left.

Since every subspace of an $\alpha$-left space is $\alpha$-left (the restriction of the almost left well-order to the subspace has the required property) we have

2.3 Corollary. If $\lambda$ and $\mu$ are ordinals with the order topology, then $\lambda \times \mu$ is $\alpha$-left if and only if $\lambda$ or $\mu$ is countable.

Thus $(\omega_1 + 1) \times (\omega_1 + 1)$ is a dispersed compact $T_2$ space which is not $\alpha$-left, showing that CH is not needed in [A2, Corollary 8].

We now begin the proof of the theorem with

2.4 Lemma. If $A$ is left-separated and $F$ is an $\alpha$-left space then $A \times F$ is $\alpha$-left.

Proof. Order $A$ with a left well-order and $F$ with an almost left well-order. It is easy to verify that the lexicographic order on $A \times F$ is an almost left well-order.

2.5 Lemma. Let $Y$ be a topological space, $\lambda$ an ordinal, $g: \lambda \rightarrow Y$ an almost left well-order of $Y$. Let $\tau$ be an infinite cardinal and $\varphi: \tau^+ \rightarrow Y$ a homeomorphism of $\tau^+$ onto $\varphi[\tau^+] = L$. Denote by $\alpha_0$ the least ordinal no greater than $\lambda$ such that $\varphi^{-1}(g[\alpha_0])$ is cofinal in $\tau^+$. Then:

(i) $\text{cof}(\alpha_0) \geq \tau^+$.

(ii) $g[\alpha_0] \cap L$ is closed in $L$.

(iii) If $\psi: \tau^+ \rightarrow Y$ is a homeomorphism of $\tau^+$ onto $M = \psi[\tau^+]$ and $M \cap L = \emptyset$, then $\psi^{-1}(g[\alpha_0])$ is not cofinal in $\tau^+$.

Proof. (i) This easy proof is left to the reader.

(ii) $g[\alpha_0 + 1] = g[\alpha_0] \cup \{g(\alpha_0)\}$ is closed in $Y$ since $g$ is an almost left well-order. If $g(\alpha_0) \notin L$, then (ii) holds. If $g(\alpha_0) \in L$, then $g(\alpha_0)$ has a neighbourhood $U$ in $L$ of cardinality no greater than $\tau$. By (i) there exists $\beta < \alpha_0$ such that $g(\gamma) \notin U$ for every $\gamma > \beta$. Since $g$ is bijective we have $g(\alpha_0) \in U \cap (\gamma - g(\beta + 1))$, this being an open subset of $L$ which misses $g[\alpha_0] \cap L$.
(iii) For $\alpha < \alpha_0$ we write $F_\alpha = g[\alpha + 1] \cap L$ and $G_\alpha = g[\alpha + 1] \cap M$; if we assume that $\psi^{-1}(g[\alpha_0])$ is cofinal in $\tau^+$, both $F_\alpha$ and $G_\alpha$ are nonempty for large $\alpha$, say for $\alpha \geq \beta_0$; for every such $\alpha$ we put $a_\alpha = \sup \psi^{-1}(F_\alpha)$, and $b_\alpha = \sup \psi^{-1}(G_\alpha)$. Since $(a_\alpha; \beta_0 \leq \alpha < \alpha_0)$ and $(b_\alpha; \beta_0 \leq \alpha < \alpha_0)$ are cofinal in $\tau^+$, it is possible to construct recursively a sequence $\beta_0 < \beta_1 < \beta_2 < \cdots$ of ordinals smaller than $\alpha_0$ such that $a_{\beta_0} < a_{\beta_1} < a_{\beta_2} < \cdots$ and $b_{\beta_0} < b_{\beta_1} < b_{\beta_2} < \cdots$. If $\beta = \sup \{\beta_n; n \in \omega\}$ then $\beta < \alpha_0$ (by (i)) and $g(\beta) \not\in L$ or $g(\beta) \not\in M$ (or both). If $g(\beta) \not\in L$ then

$$F_\beta = g[\beta + 1] \cap L = g[\beta] \cap L = \bigcup_{n \in \omega} g[\beta_n + 1] \cap L = \bigcup_{n \in \omega} F_\beta_n,$$

but $a_\beta = \sup \{a_{\beta_n}; n \in \omega\} \not\in \varphi^{-1}(F_\beta)$, which is a contradiction since $\varphi^{-1}(F_\beta)$ is closed in $\tau^+$.

2.6 Lemma. Let $X$ be a $T_1$ space with infinite tightness $\tau$. If $\tau^+ \times X$ is $\alpha$-left then $X$ is left-separated.

Proof. Let $\lambda$ be an ordinal and $g: \lambda \rightarrow \tau^+ \times X$ an almost left well-order of $\tau^+ \times X$. Given $x \in X$, $R_x = \tau^+ \times \{x\}$ is a closed copy of $\tau^+$ in $\tau^+ \times X$: let $\alpha_x$ be the smallest ordinal no greater than $\lambda$ such that $g[\alpha_x]$ is cofinal in $R_x$. By Lemma 2.5, $x \rightarrow \alpha_x$ is an injection of $X$ into $\lambda + 1$, and for every $x \in X$ the set $C_x = g[\alpha_x] \cap R_x$ is closed and cofinal in $R_x$. Well-order $X$ by defining $x < y$ if and only if $\alpha_x < \alpha_y$. We claim that $< \lambda$ is a left well-order of $X$. To prove this, let $a \in A$ and put $A = \{x \in X: x < a\}$. We wish to prove that if $p \in c_X(A)$ then $p \in A$. Since $X$ has tightness $\tau$, there exists a subset $B = \{a(\xi): \xi \in \tau\}$ of $A$ such that $p \in c_X(B)$; excluding trivial cases, we can assume that $B$ is infinite and that $a(\xi) \neq p$ for each $\xi \in \tau$. Let $G = \pi_1(G_p) \cap (\bigcap_{\xi \in \tau} \pi_1(C_{a(\xi)}))$, where $\pi_1: \tau^+ \times X \rightarrow \tau^+$ denotes the first projection. $G$ is the intersection of $\tau$ closed cofinal subsets of $\tau^+$ and hence by [J, Lemma 7.4], it is closed and cofinal in $\tau^+$. Then $G \times \{p\} \subset c(G \times \{p\})$, and $G \times \{a(\xi)\} \subset g[a_{a(\xi)}]$ for each $\xi \in \tau$. Let $\delta = \sup \{a_{a(\xi)}: \xi \in \tau\}$. Then either $\delta < \lambda$ or $\delta = \lambda$. If $\delta < \lambda$, then $g[\delta + 1]$ is a closed subset of $\tau^+ \times X$ which contains $\bigcup_{\xi \in \tau} G \times \{a(\xi)\}$, and hence also contains $G \times \{p\}$, which is cofinal in $R_p$; thus $\delta \geq \alpha_p$. If $\delta = \alpha_{a(\xi)}$ for some $\xi \in \tau$, then $\alpha_p \leq \alpha_{a(\xi)}$, and hence $p \leq a(\xi) \in A$ and so $p \in A$; if not, then $\cof \delta \leq \tau$, and hence $\delta > \alpha_p$ (since by Lemma 2.5(i), we have $\cof \alpha_p \geq \tau^+$). Thus $\alpha_p < \alpha_{a(\xi)}$ for some $\xi \in \tau$, and this implies that $p < a(\xi) \in A$, and so $p \in A$.

If $\delta = \lambda$, the above argument is valid if $\delta = \lambda = \alpha_{a(\xi)}$ for some $\xi \in \tau$. If $\delta \neq \alpha_{a(\xi)}$ for each $\xi \in \tau$, then $\delta = \lambda = \alpha_p$; but then $\cof \alpha_p = \cof \delta \leq \tau$, which is impossible by Lemma 2.5(i).

2.7 Proof of Theorem 2.1. Sufficiency (with no restriction on tightness) follows from 2.4; necessity from 2.6.

3. $\alpha$-left ordered spaces. As was pointed out in [A1], not every ordered space is $\alpha$-left; indeed, not even every zero-dimensional ordered space is $\alpha$-left, although every $\alpha$-left ordered space is certainly zero dimensional. Our aim in this section is to prove that every dispersed ordered space is $\alpha$-left.
Recall that every dispersed space $X$ can be written as a disjoint union of the form $X = \bigcup \{ D_\xi : \xi < \rho \}$ (where $\rho$ is an ordinal) and where for each $\xi < \rho$, $D_\xi$ is the set of isolated points of the closed subspace $X_\xi = \bigcup \{ D_\alpha : \alpha > \xi \}$. The least ordinal $\rho$ such that $D_\rho = \emptyset$ is called the dispersal order of $X$. It is known (see [R]), that if $X$ is compact then $\rho = \lambda + 1$ and $D_\lambda$ is finite.

3.1 Theorem. Every compact dispersed ordered space is $\alpha$-left.

Proof. First note that every totally disconnected ordered space has a base of open and closed intervals which, if the space is compact, must have first and last elements.

The proof is by induction on $\lambda = \rho - 1$, where $\rho$ is the dispersal order of $X$ (see above). If $\lambda = 0$, then $D_0 = X$ which is thus a finite Hausdorff space and hence is $\alpha$-left. We now assume that every compact dispersed ordered space of dispersal order $\delta - 1 = \mu < \lambda$ is $\alpha$-left. Since $D_\lambda = \{ p_1, \ldots, p_m \}$ for some integer $m$, we can split $X$ into a disjoint union of $m$ open and closed intervals, each containing precisely one $p_k$. We are thus reduced to the case $D_\delta = \{ p \}$, a singleton. In this case $X = [a, p] \cup [p, b]$ where $a = \min X$ and $b = \max X$; we will first show that $[a, p]$ is $\alpha$-left. Let $(U_a : a < \kappa)$ be a nested neighbourhood basis for $p$ in $[a, p]$, consisting of open and closed intervals $[a_a, p]$, indexed by a (regular) cardinal $\kappa$. For each $a \in \kappa$, let $Y_a = X - U_a = [a, a_b]$ (where $a_b$ is the immediate predecessor of $a$ in the order of $X$). By the inductive hypothesis, for each $a \in \kappa$ there exists an ordinal $\gamma_a$ which almost left separates $Y_a$. We will show that this can be done in such a way that $\alpha < \beta < \kappa$ implies that $Y_a$ is an initial segment of $Y_\beta$; this being possible, $(\bigcup_{a \in \kappa} Y_a)$ + 1 will almost left-separate $[a, p]$. Assume that this has been done for all $\beta < \alpha < \kappa$. If $\alpha$ is a nonlimit ordinal, $\alpha = \xi + 1$, then $\{ Y_\xi, Y_\alpha - U_\xi \}$ is a partition of $Y_\alpha$ into open and closed sets, and according to the inductive hypothesis, $Y_\alpha - U_\xi$ is left-separated by some ordinal $\eta_\alpha$. We can then almost left-separate $Y_\alpha$ by $\gamma_\xi + \eta_\alpha$. If $\alpha$ is a limit ordinal, $\alpha = \bigcup_{\xi \in \alpha} Y_\xi$, then we first well-order $Z_\alpha = \bigcup_{\xi \in \alpha} Y_\xi$ by $\gamma = \bigcup_{\xi \in \alpha} Y_\xi$; note that $Y_\alpha \supset Z_\alpha$ and that equality is impossible since $Y_\alpha$ is compact. Then $Y_\alpha - Z_\alpha$ is a nonempty compact ordered dispersed space which is again almost left-separated by some ordinal $\eta_\alpha$, according to the inductive hypothesis. By Lemma 1.3, the well-ordering of $Y_\alpha - Z_\alpha$ may be chosen in such a way that $\min(Y_\alpha - Z_\alpha)$ is the first element. It is easy to check that $Y_\alpha = \gamma + \eta_\alpha$ gives the desired almost left-separation of $Y_\alpha$.

In the same way we can prove that $[p, b]$ is $\alpha$-left and so it is clear that $X = [a, p] \cup \{ p \} \cup [p, b]$ is $\alpha$-left; well-order $X$ by placing $p$ first, then well-order $[a, p] \cup [p, b]$ by the ordinal sum.

3.2. Purisch [P] has proved that every dispersed suborderable space is orderable and has an ordered dispersed compactification. Thus, recalling that the $\alpha$-left property is hereditary, we have

Corollary. Every dispersed suborderable space is $\alpha$-left.

3.3. The converse result is false however: The rationals are an example of an $\alpha$-left (even left-separated) ordered space which is not dispersed. Using [A, Theorem 1] and the fact that the density and the hereditary density of an ordered space are...
equal, it is easy to show that every ordered \( \alpha \)-left space \( X \) is strongly scattered (i.e. if \( Y \subseteq X \) then \( |Y| = |\text{cl}_X(Y)| \)). This fact together with Theorem 3.1 and Theorem 3 of \([A_1]\), imply

**Corollary.** For a compact ordered space \( X \) the following are equivalent:

(i) \( X \) is \( \alpha \)-left,

(ii) \( X \) is dispersed,

(iii) \( X \) is strongly scattered.

Observe that \((\omega_1 + 1) \times (\omega_1 + 1)\) is a strongly scattered compact space which is not \( \alpha \)-left (by 2.2). On the other hand there exist strongly scattered compact \( \alpha \)-left spaces which are not orderable: \((\omega + 1) \times (\omega_1 + 1)\) is such a space. (It is not orderable since it is not hereditarily normal; it is \( \alpha \)-left by Lemma 2.1.)

**Problem.** Is every strongly scattered ordered space \( \alpha \)-left?

3.4. Telgársy [T, Theorem 9] has shown that every dispersed metric space is suborderable, hence 3.2 implies that every such space is \( \alpha \)-left. However (as the referee has pointed out to us), dispersed metric spaces are \( \sigma \)-discrete; that is, they are representable as the countable union of (mutually disjoint) closed discrete subspaces \([T, \text{Theorem 8 or S, Theorem 11}]\). Thus Lemma 1.2 implies

**Proposition.** Every dispersed metric space is left-separated.

3.5. It follows from Corollary 2.3 that since no topological product of two uncountable ordinals is \( \alpha \)-left, no such space is suborderable. This still leaves open the question of whether a product of an uncountable ordinal and a countable ordinal can be suborderable. (The fact that the product of two countable ordinals is always orderable follows from the above-mentioned results of Purisch and Telgársy.) The following theorem answers this question.

3.6. **Theorem.** Let \( \lambda, \mu \) be ordinals, \( \mu \geq \omega_1 \). Then \( \lambda \times \mu \) is suborderable if and only if \( \lambda \leq \omega. \) In this case \( \lambda \times \mu \) is actually orderable.

**Proof.** If \( \lambda \leq \omega \) then \( \lambda \) is discrete and the lexicographic order topology of \( \lambda \times \mu \), with the modification that every even column has its order reversed, is the product topology. (The modification is only needed if \( \mu \) has no last element.)

Suborderable spaces being hereditarily normal, to complete the proof it suffices to show that \( X = (\omega + 1) \times \omega_1 \) has a nonnormal subspace. Let \( Y \) be the subspace of \( X \) obtained by deleting from \( X \) all points of the form \((\omega, \lambda)\) where \( \lambda \in \omega_1 \) is a limit ordinal. We claim that \( Y \) is nonnormal. (This was pointed out to us by the referee, thus simplifying our original proof of the theorem.) Let \( A = \omega \times \Lambda, B = \{\omega\} \times N \), where \( \Lambda, N \) denote the sets of limit and nonlimit ordinals of \( \omega_1 \), respectively. Clearly \( A \) and \( B \) are closed disjoint subsets of \( Y \). If \( f \in C(Y) \) is identically zero on \( A \), then it is easy to show that there exists some \( \beta_0 \in \omega_1 \) such that \( f(n, \beta) = 0 \) for every \( n \in \omega \) and each \( \beta \geq \beta_0 \); but then \( f(\omega, \beta) = 0 \) for each \( \beta \geq \beta_0 \) which is nonlimit; that is to say, \( f \) vanishes on (infinitely many) points of \( B \). Thus \( A \) and \( B \) are not completely separated in \( Y \), and hence \( Y \) is not normal.
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SEMINARIO MATEMATICO, UNIVERSITA DI PADova, 35100 PADova, ITALy

DEPARTAMENTO DE MATEMATICAS, UNIVERSIDAD AUTONOMA METROPOLITANA, UNIDAD IZTAPALAPA, MEXICO 13, D. F., MEXICO