Approximate Identities and $H^1(\mathbb{R})$

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Abstract. Let $\varphi(x) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ be a real-valued function with $\int_{\mathbb{R}} \varphi \, dx \neq 0$.
For $y > 0$, let $\varphi_y(x) = y^{-1} \varphi(x/y)$. For $f(x) \in L^1(\mathbb{R})$ define
$$f^*_y(x) = \sup_{y > 0, t \in \mathbb{R} : |x - t| < y} |f \ast \varphi_y(t)|.$$ 
We investigate the space $H^1_{\varphi} = \{f \in L^1(\mathbb{R}) : f^*_y \in L^1(\mathbb{R})\}$.

1. Introduction. If $\varphi$ is the Poisson kernel, then $H^1_{\varphi}$ is defined to be $H^1$. Fefferman and Stein [2] showed that $H^1_{\varphi} = H^1$ for any $\varphi$ that is smooth and dies quickly at infinity; e.g. $\varphi$ can be in the Schwartz class, or Lipschitz continuous (of any order) and compactly supported. However, it is easy to show that $H^1_{\varphi} = \{0\}$ if $\varphi = \chi_{[0,1]}$, where $\chi_E$ is the characteristic function of a set $E$. G. Weiss asked whether there was an $H^1_{\varphi}$ that was nontrivial but not $H^1$. In this note, we show the following two results.

**Theorem 1.** If $H^1_{\varphi} \neq \{0\}$, then $a(x) \in H^1_{\varphi}$, where
$$a(x) = \begin{cases} 1 & 0 < x < 1, \\ -1 & -1 < x < 0, \\ 0 & \text{otherwise}. \end{cases}$$

**Theorem 2.** There exists $\varphi(x) \geq 0$ such that $H^1_{\varphi} \neq \{0\}$, $H^1_{\varphi} \neq H^1$.

As a corollary of Theorem 1, we get

**Corollary 1.** If $H^1_{\varphi} \neq \{0\}$, then $H^1_{\varphi} \cap H^1$ is dense in $H^1$.

**Comment on notation.** To distinguish the “$y$” in $\varphi_y(x) = y^{-1} \varphi(x/y)$ from the other subindices, in the following we write $(\varphi)_y$ instead of $\varphi_y$. The letter $C$ denotes various constants.

2. Proof of Theorem 1. For $f \in H^1_{\varphi}$ define
$$\|f\|_{H^1_{\varphi}} = \|f^*_y\|_{L^1}.$$ 
This norm makes $H^1_{\varphi}$ a Banach space. We use two simple facts about $\|\|_{H^1_{\varphi}}$.

**Fact 1.** If $f \in H^1_{\varphi}$, $g \in L^1$, then $f \ast g \in H^1_{\varphi}$ with
$$\|f \ast g\|_{H^1_{\varphi}} \leq \|f\|_{H^1_{\varphi}} \|g\|_{L^1}.$$
Fact 2. If \( y > 0 \) and \( f \in H^1_\psi \), then
\[
\|(f)\|_{H^1_\psi} = \|f\|_{H^1_\psi}.
\]

Let \( f \in H^1_\psi \), \( f \not\equiv 0 \) and fix \( f \). In the following part of this section, the constants \( c \) depend on this function \( f \). We may assume that \( f \) is real-valued (since \( \varphi \) is real-valued). We shall construct functions \( p_n, g_n \) \((-\infty < n < \infty)\) satisfying
\[
(1) \quad \|p_n\|_{H^1_\psi} \leq c,
\]
\[
(2) \quad \sum_{n=-\infty}^{\infty} \|g_n\|_{L^1} < +\infty,
\]
\[
(3) \quad a = \sum_{n=-\infty}^{\infty} p_n \ast g_n,
\]
where the convergence is in \( H^1_\psi \). This implies the theorem.

The construction of \( p_n \) and \( g_n \). Since \( f \not\equiv 0 \) and since \( f \) is real-valued, we may assume there exist \( r > 1 \) and \( \varepsilon > 0 \) such that
\[
|f(x)| > \varepsilon \quad \text{on} \; [-r^{-1}, r^{-1}] \cup [r, r].
\]
Let \( \psi(x) \in S(\mathbb{R}) \) be a real-valued even function such that
\[
\text{supp} \psi \subset [-r^{-1}, r^{-1}] \cup [r, r], \quad \sum_{k=-\infty}^{\infty} \psi(k\xi)^2 \equiv 1 \quad \text{for any} \; \xi \neq 0.
\]

Now we invoke Wiener’s Lemma: Let \( f_1(x), f_2(x) \in L^1(\mathbb{R}) \). If there exist an \( \varepsilon > 0 \) and an interval \( I \subset \mathbb{R} \) for which \( |f_1(\xi)| > \varepsilon, \; \xi \in I \), and \( \text{supp} f_2 \subset I \), then there is an \( h(x) \in L^1(\mathbb{R}) \) such that \( f_2(\xi) = \hat{h}(\xi) \hat{f_1}(\xi) \).

Applying Wiener’s Lemma to \( f(x) \) and \( (\hat{\psi} \chi_{(0, \infty)})^{-1} \), we get \( h_1(x) \in L^1(\mathbb{R}) \) such that
\[
\hat{\psi}(\xi) \chi_{(0, \infty)}(\xi) = \hat{h}_1(\xi) \hat{f}(\xi).
\]
Set \( \hat{h}(\xi) = \hat{h}_1(\xi) + \hat{h}_1(-\xi) \). Then \( \hat{\psi}(\xi) = \hat{h}(\xi) \hat{f}(\xi) \), and
\[
(4) \quad \|\psi\|_{H^1_\psi} \leq \|h\|_{L^1} \|f\|_{H^1_\psi} \leq c \|f\|_{H^1_\psi}.
\]

We now define
\[
p_n(x) = (\psi)_n(x), \quad g_n(x) = a \ast (\psi)_n(x).
\]
Then (1) follows from (4). By taking Fourier transforms, we see that \( a = \sum_{n=-\infty}^{\infty} p_n \ast g_n \) in \( S' \). To estimate \( \|g_n\|_{L^1} \), we divide into two cases.

Case 1. \( n \geq 0 \). We write
\[
|g_n(x)| = \left| \int_{-1}^{1} r^{-n} \psi(r^{-n}(x - t)) a(t) \, dt \right| = r^{-n} \left| \int_{-1}^{1} (\psi(r^{-n}(x - t)) - \psi(r^{-n}x)) a(t) \, dt \right| \leq cr^{-2n} \sup_{|r^{-n}x - \xi| < r^{-n}} |\psi'(\xi)| \leq cr^{-2n} R(r^{-n}x),
\]
where \( R(x) = \sup_{|x-y|<1} |\psi'(y)| \). Therefore,
\[
\|g_n\|_{L^1} \leq c r^{-2n} \int R(r^{-n}x) \, dx \leq cr^{-n}.
\]

Case 2. \( n < 0 \). We distinguish three subcases.

Subcase 1. \( |x| > 3 \).
\[
|g_n(x)| = \left| \int_{-1}^1 r^{-n}\psi(r^{-n}(x-t))a(t) \, dt \right| \leq cr^{-n}/(r^{-n}|x|)^4
\]
(\( \psi \) is rapidly decreasing). Thus, \( \int_{|x|>3} |g_n(x)| \leq cr^3n \).

Subcase 2. \( |x| \leq 3, \min(|x|, |x+1|, |x-1|) \geq r^{n/2} \). These \( x \)'s are away from the discontinuities of \( a(x) \). We have
\[
|g_n(x)| \leq \left| \int_{|x-t|<r^{n/2}} r^{-n}\psi(r^{-n}(x-t))a(t) \, dt \right| + \int_{|x-t|>r^{n/2}} \cdots \, dt
\]
The second term can be estimated as in the first subcase. The first term equals zero or it equals \( \int_{|t|>r^{n/2}} \psi(t) \, dt \) (because \( \int \psi(t) \, dt = 0 \)). This is dominated by \( cr^n \), since \( \psi \) is rapidly decreasing.

Subcase 3. \( \min(|x|, |x+1|, |x-1|) < r^{n/2} \). Here the best we can do is \( |a \ast \psi_n(x)| \leq c \). But the measure of this set is \( \leq 6r^{n/2} \).

Combining the three subcases yields for \( n < 0 \), \( \|g_n\|_{L^1} \leq cr^{n/2} \). We therefore have (2).

3. Proof of Corollary 1. It is well known that the dual space of \( H^1 \) is the space BMO (see [2]). This is the space of locally integrable functions \( h(x) \) that satisfy
\[
\sup_I |I|^{-1} \int_I |h(x) - \bar{h}_I| \, dx = \|h\|_* < \infty.
\]
The supremum is over all intervals \( I \subset \mathbb{R} \); \( \bar{h}_I \) denotes the average of \( h(x) \) over \( I \).

Clearly \( a(x) \in H^1 \). Also \( H^1 \) and \( H^1 \) are closed under translations and dilations. If \( H^1 \cap H^1 \) is not dense, then there is an \( h \in \text{BMO} \) such that \( \|h\|_* = 1 \) but \( \int h(x)g(x) \, dx = 0 \), for any \( g \in H^1 \cap H^1 \). The same must hold for any dilation or translation of \( a(x) \). This implies that \( h \) is constant and \( \|h\|_* = 0 \).

4. Proof of Theorem 2. An examination of the proof of Theorem 1 shows that it works because of the relative smoothness of \( a(x) \). In this section, we exhibit an \( H^1 \) that is not trivial or \( H^1 \), by building functions \( b(x) \in H^1 \) and \( \varphi(x) \), each of which has "large" high frequency terms in its Fourier series. The high frequencies of \( \varphi(x) \) almost cancel out when \( \varphi(x) \) is convolved with \( a(x) \), but they match up with those of \( b(x) \) to make \( b(x) \not\in H^1 \).

For \( n = 1, 2, 3, \ldots \), define
\[
\mu_n(x) = \sum_{k=1}^n \sin(2^k\pi x)\chi_{[1,2]}(x).
\]
We estimate \( |a \ast (\mu_n)_y(x)| \) as follows.
Case 1. $y < 1$.

$$|a \ast (\mu_n)_y(x)| \leq C \sum_{k=1}^{n} \left(\frac{1}{y}\right) \left(\frac{y}{2^k}\right) \leq C.$$ 

Case 2. $y > 2^n$.

$$|a \ast (\mu_n)_y(x)| \leq C \sum_{k=1}^{n} \left(\frac{1}{y}\right) \left(\frac{y}{2^k}\right) \leq C2^n/y^2.$$ 

Case 3. $1 \leq y \leq 2^n$.

$$|a \ast (\mu_n)_y(x)| \leq C \sum_{\log_2 y < k < n} \left(\frac{1}{y}\right) \left(\frac{y}{2^k}\right) + C \sum_{1 \leq k < \log_2 y} \left(\frac{1}{y}\right) \left(\frac{y}{2^k}\right) \leq C/y.$$ 

Now observe that $a \ast (\mu_n)_y(t) = 0$ if $y \leq (t - 1)/2$ or $y \leq (-t - 1)/2$. Thus

$$a_{\mu_n}^*(x) \leq \begin{cases} 
C & \text{if } |x| \leq 1, \\
C/|x| & \text{if } 1 < |x| \leq 6 \cdot 2^n, \\
C2^n/|x|^2 & \text{if } 6 \cdot 2^n < |x|.
\end{cases}$$ 

This yields $\|a_{\mu_n}^*\|_{L^1} \leq Cn$.

If $\alpha > 1$, then by

$$a \ast (\mu_n(\alpha \cdot))_y(t) = \alpha^{-1}a \ast (\mu_n)_y(\alpha t),$$

and by similar observations as above, we get

(5) $$\|a_{\mu_n(\alpha \cdot)}^*\|_{L^1} \leq Cn,$$

where $C$ does not depend on $\alpha > 1$.

Define

$$\alpha_n = 2^{2n}, \quad \eta(x) = \sum_{n=1}^{\infty} n^{-2-\varepsilon_0} \mu_n(\alpha_n x),$$

where $\varepsilon_0 > 0$ is a small number. Then, by (5) we have

(6) $$\|a_n^*\|_{L^1} \leq \sum_{n} n^{-2-\varepsilon_0} \|a_{\mu_n(\alpha_n \cdot)}^*\|_{L^1} \leq C \sum_{n} n^{-1-\varepsilon_0} < +\infty.$$

Let $\varepsilon > 0$ be a small number. Define

$$b(x) = -\sum_{k=1}^{\infty} k^{-1+\varepsilon}\sin(2^k \pi x) \chi_{[-2,-1]}(x).$$

From the fact that $b \in L^2$, $\int b \, dx = 0$ and supp $b \subset [-2,-1]$, it follows that $b \in H^1$ (see [1]).

We claim that for $n > N_\varepsilon$ and $0 \leq i \leq n/2$,

$$\left| \int b(x) \mu_n(2^{-i}(-x - 1) + 1) \, dx \right| \geq C_n n^\varepsilon.$$
This is because the left-hand side equals
\[
\left| \int b(x) \sum_{k=i+1}^{n} \sin(2^k \pi (2^{-i}(-x - 1) + 1)) \, dx + \int b(x) \sum_{k=1}^{i} \sin(2^k \pi (2^{-i}(-x - 1) + 1)) \, dx \right|
\]

The first integral equals
\[
\frac{1}{2} \sum_{k=i+1}^{n} (k - i)^{-1+\varepsilon} \geq C \varepsilon n^\varepsilon.
\]

The second integral is no larger than
\[
\| b \|_1 \left\| \sum_{k=1}^{i} \left( \sin(2^k \pi (2^{-i}(-x - 1) + 1)) \right) \right\|_\infty \leq C \sum_{k=1}^{i} 2^{-k-i} \leq C
\]
(since \( \int b \, dx = 0 \)). Thus
\[
\left| \int b(x) \mu_n(2^{-i}(-x - 1) + 1) \, dx \right| \geq C \varepsilon n^\varepsilon - C \geq C \varepsilon n^\varepsilon,
\]
if
\[
0 \leq i \leq n/2 \quad \text{and} \quad n > N_\varepsilon.
\]
Therefore, if (7) holds,
\[
b^* \eta_{2\alpha_n}(2^i - 1) = (2\alpha_n)^{-1} n^{-2-\varepsilon_0} \int b(x) \mu_n(2^{-i}(2^i - 1 - x)) \, dx \geq C \varepsilon (2\alpha_n)^{-1} n^{-\varepsilon_0+\varepsilon}.
\]
Thus, \( b^*_\eta(x) \geq C \varepsilon (2\alpha_n)^{-1} n^{-\varepsilon_0+\varepsilon} \) on \( E_{n,i} = \{ x : 2^{i-1} \alpha_n < |x| < 2^{i} \alpha_n - (2^i - 1) \} \).
Thus,
\[
\int_{E_{n,i}} b^*_\eta \, dx \geq C \varepsilon n^{-\varepsilon_0+\varepsilon},
\]
which yields, upon summing for \( 0 < i \leq n/2 \),
\[
\int_{\alpha_n < |x| < 2^n/2 \alpha_n} b^*_\eta \, dx \geq C \varepsilon n^{-\varepsilon_0+\varepsilon}.
\]
Therefore
\[
\| b^*_\eta \|_{L^1} \geq C \varepsilon \sum_n n^{-1-\varepsilon_0+\varepsilon} = +\infty,
\]
if \( \varepsilon_0 < \varepsilon \).
Take \( \nu(x) \in \mathbb{S} \) such that \( \nu(x) + \eta(x) \geq 0 \) for any \( x \in \mathbb{R} \). Then the kernel \( \varphi = \nu + \eta \) is nonnegative and \( a^*_\varphi \in L^1 \) and \( b^*_\varphi \not\in L^1 \), by (6) and (8). Thus
\[
H^1_\varphi \neq \{0\} \quad \text{and} \quad H^1_\varphi \neq H^1.
\]
REFERENCES


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