STRONGLY EXPOSED POINTS IN BOCHNER $L^p$-SPACES

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Abstract. We give necessary and sufficient conditions for vector-valued $L^p$-functions to be strongly exposed in terms of their values ($1 < p < \infty$).

In this paper we give a characterization of the strongly exposed vector-valued $L^p$-functions in terms of their values ($1 < p < \infty$). In [4] J. A. Johnson has shown that, given a finite positive measure space $(\Omega, \Sigma, \mu)$, a Banach space $V$, an $x$ in $L^p(\mu, V)$ and a $g$ in $L^q(\mu, V')$ (where $1/p + 1/q = 1$), then $x$ is strongly exposed by $g$ if the scalar function $\|x(\cdot)\|$ is strongly exposed by $\|g(\cdot)\|$ and for almost all $t$ with $x(t) \neq 0$ the value $x(t)$ is strongly exposed by $g(t)$. He left the converse as an open question, but gave a kind of supplement in the case that $V$ has RNP. It is not too obvious that the converse should hold. Namely, a similar characterization of the extremal points is valid for separable $V$ (and Borel measures on Polish spaces [3]), but not in general [2].

We are going to show that the converse does hold for Radon measures $\mu$ on locally compact spaces, no matter what properties $V$ has (Theorem 2). If $V$ is separable we may even admit arbitrary positive measures $\mu$ (Theorem 1). In contrast to the extremal point situation the proof is rather simple.

In this manner we shall have a characterization of strong exposure as a relation between elements of $L^p(\mu, V)$ and $L^q(\mu, V')$; however, this is not yet a characterization of strongly exposed points. We shall give such a characterization under additional assumptions concerning $V$ (Theorems 3 and 4).

Recall that an element $x$ of a normed space $X$ is said to be strongly exposed by an element $\varphi$ of the dual $X'$ if

(i) $\varphi x = \|\varphi\| \cdot \|x\| \neq 0$, and

(ii) each sequence $(x_n)$ in the ball with radius $\|x\|$, such that $\varphi x_n$ converges to $\varphi x$, converges to $x$ in norm.

For arbitrary functions $x: \Omega \to V$ and $g: \Omega \to V'$ let us denote the functions $t \mapsto \|x(t)\|$, $\|g(t)\|$ and $g(t)x(t)$ by $|x|$, $|g|$ and $\langle x, g \rangle$, respectively. $\chi_A$ is the characteristic function of the subset $A$ of $\Omega$, and $v$ is the constant function with value $v$. $B(v, \varepsilon)$ denotes the closed ball with center $v$ and radius $\varepsilon$. 

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Before stating the theorems we make a few observations. Assume \( g \in L^q(\mu, V) \) strongly exposes \( x \in L^p(\mu, V) \). Then from
\[
\|g\| \cdot \|x\| = gx = \int \langle x, g \rangle d\mu \leq \int |g| \cdot |x| d\mu \leq \|g\| \cdot \|x\|,
\]
we deduce \( \langle x, g \rangle = |g| \cdot |x| \) a.e. and, consequently, that \( \chi_A g \) strongly exposes \( \chi_A x \) for all measurable \( A \) with \( \chi_A x \neq 0 \). Thus, in order to prove the theorems below, knowing that \( x \) and \( g \) vanish outside a suitable \( \sigma \)-finite set, we may assume without loss of generality that \( \mu \) is finite. For the same reason we may assume that \( |x| \) and \( |g| \) are strictly positive. We may also assume w.l.o.g. that the scalars are real, since for complex scalars \( \varphi \) strongly exposes \( x \) if and only if the real part \( \text{Re} \circ \varphi \) of \( \varphi \) strongly exposes \( x \) in the underlying real space.

**Theorem 1.** Let \((\Omega, \Sigma, \mu)\) be a positive measure space, \( V \) a separable Banach space, \( 1 < p < \infty \), \( x \in L^p(\mu, V) \) and \( g \in L^q(\mu, V) \). Then \( g \) strongly exposes \( x \) if and only if
\[
|g| \text{ strongly exposes } |x| \quad \text{(i.e. } |g|/\|g\| = (|x|/\|x\|)^{p-1} \text{) and for almost all } t \in \Omega,
\]
\( g(t) \) strongly exposes \( x(t) \) or \( g(t) = 0 = x(t) \).

**Proof.** The “if” part is Theorem 1 in [4], where the finiteness of \( \mu \) is an unnecessary restriction. Now let \( g \) strongly expose \( x \). From \( \int |g| \cdot |x| d\mu = \|g\| \cdot \|x\| \) and the uniform convexity of \( L^p(\mu) \), it is clear that \( |g| \) strongly exposes \( |x| \). By the preceding remarks we can assume that \( |x| \) and \( |g| \) are strictly positive, \( \mu \) is finite and the scalars are real. For \( t \in \Omega \) and \( n \in \mathbb{N} \) define the slices
\[
S(t, n) := \{v \in V | \|v\| < |x|(t), g(t)v > (1 - 1/n) \cdot |g|(t) \cdot |x|(t)\},
\]
\[
d(t, n) := \min \{\epsilon > 0 | S(t, n) \subset B(x(t), \epsilon)\},
\]
and
\[
e(t) := \inf \{d(t, n) | n \in \mathbb{N}\}.
\]
Observe that \( g(t) \) strongly exposes \( x(t) \) if and only if \( e(t) = 0 \).

First we want to show that \( e \) is a measurable function. By the definition of \( e \) it is sufficient to show that the functions \( d(\cdot, n) \) are measurable, i.e. the sets \( \{t | d(t, n) > \delta\} \) are measurable for all \( \delta > 0 \). Fix such a \( \delta \). For \( v \in V \) define \( A_{n,v} := \{t | \|v\| < |x|(t), g(t)v > (1 - 1/n) \cdot |g|(t) \cdot |x|(t)\} \). \( A_{n,v} \) is measurable since all the functions involved are measurable (w.l.o.g. \( \mu \) is a complete measure). Now let \( D \) be a countable dense subset of \( V \). Observe that \( d(t, n) > \delta \) if and only if \( t \in A_{n,v} \) for a suitable \( v \in D \). Thus we conclude that \( \{t | d(t, n) > \delta\} \) is measurable as a countable union of measurable sets.

It remains to show that \( \{t | e(t) > 0\} \) has measure zero. Assume the contrary. Then there is a \( \delta > 0 \) such that \( A := \{t | e(t) > \delta\} \) has positive measure. We want to construct a sequence \( (y_n) \) in \( L^p(\mu, V) \) such that \( y_n - x \geq \delta \) on \( A \), \( |y_n| \leq |x| \) and \( \langle y_n, g \rangle \geq (1 - 1/n) \cdot |g| \cdot |x| \) a.e. Then \( \|y_n\| \leq \|x\| \), \( \|y_n - x\|^p \geq \mu(A) \cdot \delta^p \) and \( g y_n \geq (1 - 1/n) \cdot |g| \cdot \|x\| \), which means that \( g \) does not strongly expose \( x \), a contradiction. To this end let \( n \in \mathbb{N} \) and define the sets \( A_{n,v} \) as above. Then
\[
A \subset \{t | d(t, n) > \delta\} = \bigcup_{v \in D} A_{n,v}.
\]
Hence \( A = \bigcup_{m=1}^{\infty} B_m \), where each \( B_m \) is a measurable set contained in some \( A_{n,v_m} \), \( v_m \in D \). Obviously \( y_n := \chi_{\Omega \setminus A} x + \sum_{m=1}^{\infty} v_m \cdot \chi_{B_m} \) has the desired properties. □
Theorem 2. Let $\mu$ be a Radon measure on a locally compact space $\Omega$, $V$ any Banach space, $1 < p < \infty$, $x \in L^p(\mu, V)$ and $g \in L^q(\mu, V')$. Then $g$ strongly exposes $x$ if and only if $|g|$ strongly exposes $|x|$ and for almost all $t \in \Omega$, $g(t)$ strongly exposes $x(t)$ or $g(t) = 0 = x(t)$.

Proof. We proceed as in the proof of Theorem 1; we have to show that $\{t \mid e(t) > 0\}$ is a null set. By Lusin's theorem the restrictions of $x$ and $g$ to suitable compact subsets, whose complements have arbitrarily small measures, are continuous. Thus we may assume w.l.o.g. that $x$ and $g$ are continuous on $\Omega$. Consequently the sets $A_{n,v}$ are open, and so is their union $\bigcup_{v \in V} A_{n,v} = \{t \mid d(t, n) > \delta\}$. This shows the measurability of $e$.

In order to verify that $\{t \mid e(t) > 0\}$ is a null set, replace the set $A$ in the proof of Theorem 1 by a compact subset with positive measure ($A \subset \{t \mid e(t) > \delta\}$, $A$ compact, $\mu(A) > 0$) and proceed as above. Then by its compactness $A$ is contained in a finite union of sets $A_{n,v}$, and the functions $y_n$ defined analogously form the desired sequence. \hfill $\square$

Since the dual of $L^p(\mu, V)$ is $L^q(\mu, V')$ if $V'$ has RNP, the following is an immediate corollary from the preceding theorems.

Theorem 3. Let $V'$ have RNP. Assume that $\mu$ is a Radon measure or $V$ is separable. Then for each strongly exposed $x \in L^p(\mu, V)$ almost all values $x(t)$ are strongly exposed or zero.

As mentioned before this is not yet a characterization of strongly exposed points in $L^p(\mu, V)$. Namely, given an $x \in L^p(\mu, V)$ such that almost all $x(t)$ are strongly exposed by some $g(t) \in V'$, we do not know whether the $g(t)$'s fit together in a measurable way. We do, however, if $V$ is smooth.

Theorem 4. Let $V$ be smooth, $\mu$ arbitrary. Then each $x \in L^p(\mu, V)$ with $x(t)$ strongly exposed or zero a.e. is strongly exposed.

Proof. W.l.o.g. $\|x\| = 1$. We may choose an $A \in \Sigma$ s.t. $\chi_A x = 0$ and, for all $t \not\in A$, $x(t)$ is strongly exposed by some norm 1 functional $g_0(t)$, and a sequence of simple functions $x_n$, vanishing on $A$ and taking only strongly exposed values outside $A$ (namely, certain $x(t)$'s), such that $x_n(t) \to x(t)$ everywhere. Put $g_0(t) := 0$ for $t \in A$. Since the support mapping $v \mapsto$ norm 1 functional supporting the unit ball in $v/\|v\|$ is norm-$\sigma(V', V)$-continuous on the unit sphere [1, p. 22], hence everywhere on $V \setminus \{0\}$, $g_0$ is the weak-* limit of a sequence of simple functions, hence weak-* measurable. From this it is easy to see that $\langle y, g_0 \rangle$ is measurable for all $y \in L^p(\mu, V)$ and that $\varphi y := \int \langle y, x \rangle x^p g_0^* \, d\mu$ defines a linear functional on $L^p(\mu, V)$ with $\varphi x = 1$ and $\|\varphi\| = 1$. Although $g := |x|^p g_0$ need not be Bochner measurable, $|g|$ is in $L^q(\mu)$ and the proof of [4, Theorem 1] shows that $\varphi$ strongly exposes $x$. \hfill $\square$

Added in proof. We can dispose of the RNP requirement in Theorem 3.

Theorem 3'. Assume that $\mu$ is a Radon measure or $V$ is separable. Then for each strongly exposed $x \in L^p(\mu, V)$ almost all values $x(t)$ are strongly exposed or zero.
This is a consequence of the facts that 1. any functional \( \varphi \) on \( L^p(\mu, V) \) may be regarded as a weak-* measurable function \( g: \Omega \rightarrow V' \) such that the upper integral \( \int|g|^q d\mu \) equals \( \|\varphi\|^q \) and \( \varphi y = \int \langle y, g \rangle d\mu \) [5, p. 97], and 2. that Theorems 1 and 2 are valid also for these \( g \) (where the measurability of \( |g| \) is implicit in \( "|g| \) strongly exposes \( |x| \)\)). In order to verify the second fact recall that the proof of [4, Theorem 1] shows the sufficiency. If in the paragraph preceding Theorem 1 we replace \( |g| \) by a measurable function \( f \gg |g| \) such that \( |g|^q \) and \( f^q \) have the same (upper) integral, the arguments of this paragraph prove that \( \langle x, g \rangle = |g| \cdot |x| = f \cdot |x| \) a.e. and \( f = 0 \) a.e. on \( \{t \mid |x| (t) = 0\} \). Consequently \( |g| \) is measurable. But then (again assuming w.l.o.g. that \( |x| \) and \( |g| \) are strictly positive) the proofs of Theorems 1 and 2 work, since we only needed that the functions \( \langle y, g \rangle (y \in V) \) and \( x \) and \( |g| \cdot |x| \) are measurable. □

Remark. Although each strongly exposing \( g \) has a measurable norm function \( |g| \), it is not true that \( g \) itself is measurable: as \( L^p(\mu, l^1) \) has RNP, the strongly exposing functionals are dense in its dual which contains \( L^q(\mu, l^\infty) \) as a proper closed subspace because \( l^\infty \) lacks RNP (\( \mu \) not purely atomic).

BIBLIOGRAPHY


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