LOGICS WITH GIVEN CENTERS AND STATE SPACES

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Abstract. Let \( B \) be a Boolean algebra and let \( K \) be a compact convex subset of a locally convex topological linear space. Then there exists a logic with the center Boolean isomorphic to \( B \) and with the state space affinely homeomorphic to \( K \).

Introduction. In the quantum logic approach to the foundations of quantum mechanics, one identifies the event structure of a system with an orthomodular partially ordered set \( L \) (called usually a logic). The set of states is then represented by the set \( \mathcal{S}(L) \) of all probability measures on \( L \) (see [4, 7]). It can be shown that \( \mathcal{S}(L) \) is a compact convex set and conversely, it was proved by F. W. Shultz [6] that any compact convex subset of a locally convex topological linear space is affinely homeomorphic to \( \mathcal{S}(L) \) for a logic \( L \).

The center \( C(L) \) of a logic \( L \) is the subset of \( L \) consisting of all "absolutely compatible" elements. It is known that the center of \( L \) is a Boolean algebra (see [1, 4]). Obviously, any Boolean algebra is the center of a logic.

Let us now consider the center and the state space simultaneously. The question is if for any Boolean algebra \( B \) and any compact convex subset of a LCTLS there exists a logic \( L \) such that \( C(L) = B \) and \( \mathcal{S}(L) = K \). We answer the question in the affirmative. In the construction we use, among other tools, the result of Shultz [6] and the technique of R. Greechie [2] for constructing orthomodular posets.

Notions. Results. Let us first review the basic definitions and state some auxiliary propositions.

Definition 1. A logic is a set \( L \) endowed with a partial ordering \( \leq \) and a unary operation \( ' \) such that:

(i) \( 0, 1 \in L \);
(ii) \( a \leq b \Rightarrow b' \leq a' \) for any \( a, b \in L \);
(iii) \( (a')' = a \) for any \( a \in L \);
(iv) \( a \lor a' = 1 \) for any \( a \in L \);
(v) \( \bigvee_{n=1}^{\infty} a_n \) exists in \( L \) whenever \( a_n \in L, a_n \leq a_n' \) for \( n \neq k \);
(vi) \( b = a \lor (b \land a') \) whenever \( a, b \in L, a \leq b \).

In the sequel, we shall reserve the symbol \( L \) for logics. One can prove easily that if \( a, b \in L, a \leq b' \) then \( a \lor b, a \land b \) exists in \( L \).

Definition 2. Two elements \( a, b \in L \) are called compatible if there are three elements \( c, d, e \in L \) such that \( c \leq d' \), \( d \leq e' \), \( e \leq c' \) and \( a = c \lor d, b = c \lor e \).
Definition 3. An element \( a \in L \) is called central if \( a \) is compatible to any element of \( L \). We denote by \( C(L) \) the set of all central elements of \( L \) and call \( C(L) \) the center of \( L \).

Proposition 1. The set \( C(L) \) with the operations ', \( \lor \), \( \land \) inherited from \( L \) is a Boolean algebra.

Proof. See [1, 4].

Definition 4. Let \( \{ L_\alpha \mid \alpha \in I \} \) be a collection of logics. Denote by \( \prod_{\alpha \in I} L_\alpha \) the ordinary Cartesian product of the sets \( L_\alpha \) and endow the set \( \prod_{\alpha \in I} L_\alpha \) with the relation \( \leq \) and the unary operation ' as follows. If \( k = \{ k_\alpha \mid \alpha \in I \} \in \prod_{\alpha \in I} L_\alpha \) and \( h = \{ h_\alpha \mid \alpha \in I \} \in \prod_{\alpha \in I} L_\alpha \), then \( k \leq h \) (resp. \( k' = h \)) if and only if \( k_\alpha \leq h_\alpha \) (resp. \( k'_\alpha = h_\alpha \)) for any \( \alpha \in I \). The set \( \prod_{\alpha \in I} L_\alpha \) with the above defined \( \leq \), ' is called the product of the collection \( \{ L_\alpha \mid \alpha \in I \} \).

Proposition 2. Let \( \{ L_\alpha \mid \alpha \in I \} \) be a collection of logics. Then \( \prod_{\alpha \in I} L_\alpha \) is a logic. If \( C(L_\alpha) = \{0, 1\} \) for any \( \alpha \in I \) then \( C(\prod_{\alpha \in I} L_\alpha) \) is Boolean isomorphic to the Boolean algebra of all subsets of \( I \).

Proof. See [3, 5].

Definition 5. A state on a logic \( L \) is a mapping \( s : L \to (0, 1) \) such that:

(i) \( s(1) = 1 \);

(ii) if \( a, b \in L \), \( a \leq b \) then \( s(a \lor b) = s(a) + s(b) \).

Let us denote by \( S(L) \) the set of all states on \( L \). By a result of F. W. Shultz [6], any compact convex subset of a LCTLS equals, up to an affine homeomorphism, \( S(L) \) for a logic \( L \) (and vice versa, which is obvious).

Definition 6. A logic \( L \) is called poor (resp. rigid) if \( S(L) = \emptyset \) (resp. \( | S(L) | = 1 \)).

It is known (see [2, 6]) that there are (finite) examples of poor and rigid logics.

Proposition 3. Suppose that \( L \) is a poor logic. Put \( L_\alpha = L \) for any \( \alpha \in I \). Then \( \prod_{\alpha \in I} L_\alpha \) is also a poor logic.

Proof. Take the mapping \( f : L \to \prod_{\alpha \in I} L_\alpha \) such that \( f(k) = (k, k, k...) \) for any \( k \in L \). If \( s \in S(\prod_{\alpha \in I} L_\alpha) \) then \( sf \in S(L) \).

Definition 7. A mapping \( f : L_1 \to L_2 \) is called an embedding if \( f \) is injective and the following requirements are satisfied.

(i) \( f(1) = 1 \);

(ii) \( f(a') = f(a)' \) for any \( a \in L_1 \);

(iii) \( a \leq b \) if and only if \( f(a) \leq f(b) \);

(iv) if \( a \leq b \) then \( f(a \lor b) = f(a) \lor f(b) \).

Proposition 4. Let \( K \) be a compact convex subspace of a LCTLS. Take the logic \( L_1 \) constructed in [6, Theorem, p. 321]. Thus \( S(L_1) = K \) and moreover, \( C(L_1) = \{0, 1\} \) and \( L_1 \) can be embedded into a poor logic \( L_2 \) with \( C(L_2) = \{0, 1\} \).

Proof. We must assume here that the reader is well acquainted with the paper [6] and with the Greechie representation of logics (see [2]). It follows immediately from
the construction of [6] that $C(L_1) = \{0, 1\}$ (see e.g. the plan of the construction, p. 321). Further, let us consider the Greechie diagram $D_1$ of $L_1$ and the Greechie diagram $D$ of a finite poor logic $L$ exhibited in [2]. Let us choose "points" $d_1 \in D_1$, $d_2 \in D$ such that $d_1$, $d_2$ belong to exactly one Boolean block of $D_1$, $D$. Form a new Greechie diagram $D_2$ by taking the union $D_1 \cup D$ and then "identifying" the points $d_1$, $d_2$. The diagram $D_2$ then represents the required logic $L_2$.

We are now ready to prove our result.

**Theorem.** Let $B$ be a Boolean algebra and let $K$ be a compact convex subset of a LCTLS. Then there exists a logic $L$ such that $C(L)$ is Boolean isomorphic to $B$ and $S(L)$ is affinely homeomorphic to $K$.

**Proof.** We may suppose that $B$ is a Boolean algebra of subsets of a set $A$. Take a logic $M$ such that $C(M) = \{0, 1\}$, $S(M) = K$ and denote by $P$ the poor extension of $M$ (Proposition 4). Take a point $a \in A$ and write $L_c = P$ if $c \in A - \{a\}$, $L_a = M$. Consider the logic $R = \prod_{d \in A} L_d$. The desired logic $L$ will now be obtained as a sublogic of $R$. Let us describe the elements of $L$. An element $r \in R$ belongs to $L$ if and only if there exists a finite partition $\mathcal{P}$ of $A$, $\mathcal{P} = \{A_i\}_{i = 1, 2, \ldots, n}$ such that $A_i \in B$ for any $i$, $1 \leq i \leq n$, and $r_p = r_q$ as soon as $(p, q) \in A_i$ for an index $i$, $1 \leq i \leq n$. We are to show that $L$ is a logic with $C(L) = B$ and $S(L) = K$.

Obviously, $1 \in L$ and if $k \in L$ then $k' \in L$. If $k, h \in L$, $k \geq h$ then $k = h \lor (k \land h')$. Indeed, if $\mathcal{P}, \mathcal{R}$ are partitions corresponding to $k$, $h$ then $\mathcal{P} \cap \mathcal{R}$ is the partition corresponding to $k' \land h$. The rest is obvious. Thus $L$ is a logic.

Further, since $C(L_d) = \{0, 1\}$ for any $d \in A$ then any central element of $L$ must have only the elements 0, 1 for the coordinates. One can check easily that $k = \{k_d \mid d \in A\}$, where any $k_d$ is either 0 or 1, belongs to $L$ if and only if $D = \{d \mid k_d = 1\} \in B$. Consequently, $C(L) = B$.

It remains to prove that $S(L) = K$. Since $S(M) = K$, it suffices to show that there is an affine homeomorphism $g: S(L) \to S(M)$. Assume that $s \in S(L)$ For any $m \in M$, denote by $k^m$ the element of $L$ which has $m$ for all its coordinates. Define $g(s)$ such that $g(s)(m) = s(k^m)$. We need to show that $g$ is injective.

Let us suppose that $g(s_1) = g(s_2)$. Take an element $k \in L$ and assume that $\mathcal{P}$ is the partition corresponding to $k$. Let $A_i$ be such a set of $\mathcal{P}$ that $a \in A_i$. Denote by $h = \{h_d \mid d \in A\}$ the element of $L$ with $h_d = 0$ if $d \in A_i$, $h_d = 1$ otherwise. It follows from Proposition 3 that $s_1(k \land h) = s_2(k \land h) = 0$. Since $g(s_1) = g(s_2)$, we see, again applying Proposition 3, that $s_1(k) = s_1(k \land h') = s_2(k \land h') = s_2(k)$. Hence the mapping $g: S(L) \to S(M)$ is injective and the proof is complete.

Let us state explicitly the following special corollary.

**Corollary.** Given a Boolean algebra $B$, there exists a poor (resp. rigid) logic $L$ such that $C(L) = B$.

Let us observe in conclusion that a similar method yields an analogous result for $\sigma$-complete logics and $\sigma$-additive states. Naturally, the center then cannot be arbitrary since there are Boolean $\sigma$-algebras without any $\sigma$-additive state.
Theorem. Let $B$ be a Boolean $\sigma$-algebra of subsets of a set and let $K$ be a compact convex subset of a LCTLS. Then there is a $\sigma$-complete logic $L$ such that $C(L)$ is Boolean $\sigma$-isomorphic to $B$ and the space of $\sigma$-additive states on $L$ is affinely homeomorphic to $K$.

References


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