COMPLETE HYPERSURFACES WITH RS = 0 IN $E^{n+1}$

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Abstract. A locally symmetric Riemannian manifold satisfies $RR = 0$ and in particular $RS = 0$. The purpose of this paper is to show that the conditions $RR = 0$ and $RS = 0$ are equivalent for complete hypersurfaces in $E^{n+1}$ and to give by $RS = 0$ some characterizations of locally symmetric hypersurfaces in $E^{n+1}$.

If a Riemannian manifold $M$ is locally symmetric, then its curvature tensor $R$ satisfies

(0.1) $R(X, Y) \cdot R = 0$

for any tangent vectors $X$ and $Y$, where the endomorphism $R(X, Y)$ operates on $R$ as a derivation of the tensor algebra at each point of $M$. Nomizu [5] proved the following:

Let $M$ be a connected and complete Riemannian $n$-manifold which is isometrically immersed in a Euclidean space $E^{n+1}$ so that the type number $k(x) \geq 3$ at least at one point $x$. If $M$ satisfies condition (0.1), then it is of the form $M = S^k \times E^{n-k}$, where $S^k$ is a hypersurface in a Euclidean subspace $E^{k+1}$ of $E^{n+1}$ and $E^{n-k}$ is a Euclidean subspace orthogonal to $E^{k+1}$.

Let $S$ be the Ricci tensor of $M$. Then the condition (0.1) implies in particular

(0.2) $R(X, Y) \cdot S = 0$

for any tangent vectors $X$ and $Y$. Then Tanno [8] showed the following results:

(1) For hypersurfaces in $E^{n+1}$ with the positive scalar curvature, the conditions (0.1) and (0.2) are equivalent. By using (1),

(2) Let $M$ be a connected and complete Riemannian $n$-manifold which is isometrically immersed in a Euclidean space $E^{n+1}$ so that the type number $k(x) \geq 3$ at least at one point $x$. If $M$ satisfies the condition (0.2) and has the positive scalar curvature, then it is of the form $M = S^k \times E^{n-k}$.

These were generalized by Ryan [7] in the case of hypersurfaces with the nonnegative scalar curvature or constant scalar curvature.

The purpose of this paper is to prove the following

Theorem. For complete hypersurfaces in $E^{n+1}$, the conditions (0.1) and (0.2) are equivalent.
1. **Lemmas.** We shall assume that $M$ is oriented and that the second fundamental form $A$ has three distinct eigenvalues $\lambda(x)$, $\mu(x)$ and $0$ which have constant multiplicities $p (\geq 2)$, $q (\geq 2)$ and $n - p - q (> 0)$, respectively. We define three distributions on $M$ as follows:

\[ T_\lambda(x) = \{ X \in T_x(M) \mid AX = \lambda(x)X \}, \]

\[ T_\mu(x) = \{ X \in T_x(M) \mid AX = \mu(x)X \}, \]

\[ T_0(x) = \{ X \in T_x(M) \mid AX = 0 \}. \]

We have $T_x(M) = T_\lambda(x) + T_\mu(x) + T_0(x)$ (direct sum). For any $Z \in T_x(M)$, $Z_{\lambda}$, $Z_{\mu}$, $Z_0$ will denote the components of $Z$ in $T_\lambda(x)$, $T_\mu(x)$ and $T_0(x)$, respectively. Then we can easily show the following [2, 5]

**Lemma 1.** $T_\lambda$, $T_\mu$ and $T_0$ are differentiable and involutive.

From $p \geq 2$ (resp. $q \geq 2$) we have [5]

**Lemma 2.** If $X$ belongs to $T_\lambda$ (resp. $T_\mu$), then $X \cdot \lambda = 0$ (resp. $X \cdot \mu = 0$).

Now, let $X \in T_\lambda$, $Y \in T_0$ and computing both sides of the Codazzi equation, we get [2]

**Lemma 3.** (i) If $X \in T_\lambda$ and $Y \in T_0$, then $\nabla X Y_\lambda = -((Y \cdot \lambda) / \Lambda)X$, where $\nabla$ denotes the covariant differentiation for the Riemannian connection on $M$.

(ii) If $Y \in T_0$, then $\nabla Y(T_0) \subset T_0$.

Similarly, we have [4]

**Lemma 4.** If $X \in T_\lambda$ and $Y \in T_\mu$, then

\[ \nabla X Y_\lambda = -((Y \cdot \lambda) / (\lambda - \mu))X \quad \text{and} \quad \nabla Y X_\mu = ((X \cdot \mu) / (\lambda - \mu))Y. \]

The following lemma is basic.

**Lemma 5 (Cartan [1]).** Let $M$ be a hypersurface in a space $\tilde{M}^{n+1}(c)$ of constant curvature $c$, $c \leq 0$, whose principal curvatures are constant. Then at most two of them are distinct.

By Lemma 1, around each point $x$ of $M$ we can choose an orthonormal frame $\{ X_1, \ldots, X_p, Y_{p+1}, \ldots, Y_{p+q}, Z_{p+q+1}, \ldots, Z_n \}$ such that $\{ X_1, \ldots, X_p \}$, $\{ Y_{p+1}, \ldots, Y_{p+q} \}$ and $\{ Z_{p+q+1}, \ldots, Z_n \}$ are bases for the distributions $T_\lambda$, $T_\mu$ and $T_0$ respectively. Hereafter, we shall use indices $a$, $b$, $c$ for the range $1, \ldots, p$; $i$, $j$, $k$ for $p + 1, \ldots, p + q$ and $r, s, t$ for $p + q + 1, \ldots, n$. From the Codazzi equation we have

\[ g((\nabla_{X_a} A) Y_i, Z_r) = g((\nabla_{Y_i} A) Z_r, X_a) = g((\nabla_{Z_a} A) X_a, Y_i), \]

i.e.,

\[ (1.1) \quad \mu g(\nabla_{X_a} Y_i, Z_r) = -\lambda g(\nabla_{Y_i} Z_r, X_a) = (\lambda - \mu) g(\nabla_{Z_a} X_a, Y_i), \]

for all $a$, $i$, $r$ and, hence,

\[ (1.2) \quad g(\nabla_{Y_i} Z_r, X_a)g(\nabla_{X_a} Y_i, Z_r) + g(\nabla_{X_a} Y_i, Z_r)g(\nabla_{Z_a} X_a, Y_i) + g(\nabla_{Z_a} X_a, Y_i)g(\nabla_{Y_i} Z_r, X_a) = 0, \]
unless
\[ g(\nabla_{Y_i} Z_r, X_a)g(\nabla_{X_a} Y_i, Z_r)g(\nabla_{Z_r} X_a, Y_i) = 0. \]

In terms of Lemma 3(ii) we get
\[ g(\nabla_{Z_r} Z_r, X_a) = g(\nabla_{Z_r} Z_r, Y_i) = 0. \]

Now, we assume that \((p - 1)\lambda + (q - 1)\mu = 0\). Differentiating \((p - 1)\lambda + (q - 1)\mu = 0\) in each direction, we get
\[ g(\nabla_{X_a} X_a, Y_i) = g(\nabla_{Y_i} Y_i, X_a) = 0, \]
and
\[ g(\nabla_{X_a} X_a, Z_r) = g(\nabla_{Y_i} Y_i, Z_r) \]
for all \(a, i, r\). Then (1.2), (1.3), Lemmas 3 and 4 give
\[ 0 = g(R(X_a, Z_r)Z_r, X_a) \]
\[ = Z_r g(\nabla_{X_a} X_a, Z_r) - \sum g(\nabla_{Z_r} Z_r, Z_r)g(\nabla_{X_a} X_a, Z_r) \]
\[ -g(\nabla_{X_a} X_a, Z_r)^2 - 2\sum g(\nabla_{Y_i} Y_i, Z_r)g(\nabla_{Z_r} X_a, Y_i), \]
(1.5)
\[ \lambda \mu = g(R(X_a, Y_i) Y_i, X_a) \]
\[ = -\sum g(\nabla_{X_a} X_a, Z_r)g(\nabla_{Y_i} Y_i, Z_r) - 2\sum g(\nabla_{X_a} Y_i, Z_r)g(\nabla_{Y_i} Z_r, X_a), \]
(1.6)
\[ 0 = g(R(X_a, Y_i) Y_j, X_a) = -2\sum g(\nabla_{X_a} Y_j, Z_r)g(\nabla_{Y_j} Z_r, X_a), \]
for \(i \neq j\),
\[ 0 = g(R(X_a, Y_i) Y_i, X_b) = -2\sum g(\nabla_{Y_i} X_a, Z_r)g(\nabla_{X_b} Z_r, Y_i), \]
(1.7)
for \(a \neq b\). Hence by a similar argument to Proposition 2.1 of [3] we obtain

**Lemma 6.** If \((p - 1)\lambda + (q - 1)\mu = 0\) and \(M\) is complete, then \(g(\nabla_{X_a} Y_i, Z_r) \neq 0\) for some \(a, i, r\).

**Proof.** Suppose \(g(\nabla_{X_a} Y_i, Z_r) \equiv 0\) for all \(a, i, r\). Since a leaf of \(\mathcal{K}\) of \(T_0\) is totally geodesic and complete [2,3], choosing \(Z_r\) as a unit tangent vector field along a geodesic \(L(s)\) of \(\mathcal{K}\), we can write (1.5) as
\[ Z_r g(\nabla_{X_a} X_a, Z_r) = g(\nabla_{X_a} X_a, Z_r)^2. \]
(1.5)'

Note that we may assume \(\lambda > 0\) and that
\[ g(\nabla_{X_a} X_a, Z_r) = Z_r(\log \lambda) \]
is considered as a function on \(L(s)\). Hence \(g(\nabla_{X_a} X_a, Z_r) \equiv 0\) or \(g(\nabla_{X_a} X_a, Z_r) \equiv (s_0 - s)^{-1}\) for some constant \(s_0\). Combining Lemmas 2–5, (1.3) and (1.4), the former cannot occur. If the latter holds, then \(g(\nabla_{X_a} X_a, Z_r)\) is not defined at \(s = s_0\), which contradicts completeness.

From (1.7) and (1.8), we get \(g(\nabla_{X_a} Y_i)_0, (\nabla_{X_a} Y_j)_0 = 0\) and \(g((\nabla_{X_a} Y_i)_0, (\nabla_{X_a} Y_j)_0) = 0\) for \(i \neq j\) and \(a \neq b\), using (1.1). On the other hand, \(|(\nabla_{X_a} Y_i)_0| = |(\nabla_{X_a} Y_j)_0|\) for
all \(a, i, b, j\) follows from (1.6). Let \(\mathcal{C}_a\) be the \(q\)-dimensional subspace of \(T_0\) spanned by \((\nabla X_a Y_i)_0, i = p + 1, \ldots, p + q\), and \(\mathcal{D}_i\) be the \(p\)-dimensional subspace of \(T_0\) spanned by \((\nabla X_a Y_i)_0, a = 1, \ldots, p\), on an open subset \(G = \{x \in M; \Sigma_r g(\nabla X_a Y_i, Z_r)^3 \neq 0 \text{ at } x\}\). Then we have [3]

**Lemma 7.** Under the assumption of \((p - 1)\lambda + (q - 1)\mu = 0\), \(\mathcal{C}_a = \mathcal{D}_i\) for all \(a, i\).

2. Proof of Theorem. Our conditions (0.1) and (0.2) reduce respectively to

\[(2.1) \quad \lambda_i \lambda_j X_k (\lambda_i - \lambda_j) = 0\]

and

\[(2.2) \quad \lambda_i \lambda_j (\lambda_i - \lambda_j)(\text{trace } A - \lambda_i - \lambda_j) = 0\]

[6, 7]. Assume (2.2). If the type number \(k(x) \leq 2\) for any \(x \in M\), then (2.1) is automatically satisfied. Hence we may suppose the type number \( \geq 3\) at some point, say, \(0 \in M\). Let \(\lambda\) and \(\mu\) be distinct nonzero principal curvatures at \(0\). If \(\nu\) is a principal curvature distinct from \(\lambda\) and \(\mu\), we have

\[\nu(\text{trace } A - \lambda - \nu) = 0, \quad \nu(\text{trace } A - \mu - \nu) = 0.\]

Since \(\lambda \neq \mu\) we must conclude that \(\nu = 0\). But if this is true, then \(\text{trace } A = \lambda + \mu\).

On the other hand, \(\text{trace } A = p\lambda + q\mu\), where \(p\) and \(q\) are the appropriate multiplicities. Thus, \((p - 1)\lambda + (q - 1)\mu = 0\) and \(p\) and \(q\) are greater than 1, since \(k(0) \geq 3\).

If \(p + q = n > 2\), the standard argument of [6] shows that \(\lambda\) and \(\mu\) are constant near \(0\). Thus, \(\lambda\mu = 0\), which implies a contradiction. Thus, at most two principal curvatures are distinct and (2.1) holds. Hence we may assume \(p + q < n\).

Let \(W = \{x | k(x) \geq 3\}\), which is an open set. Let \(W_0\) be the connected component of \(0\) in \(W\). By the above argument we see that either

\[(2.3) \quad A\text{ has only one eigenvalue } \lambda,\]

\[(2.4) \quad A\text{ has two distinct principal curvatures } \lambda \text{ and } 0, \text{ or}\]

\[(2.5) \quad A\text{ has three distinct principal curvatures } \lambda, \mu \text{ and } 0\]

holds at \(0\) and then on \(W_0\). If we assume (2.3) on \(W_0\), then \(W_0\) is umbilic. Hence \(\lambda\) is constant on \(W_0\). Next, assume that (2.5) holds on \(W_0\). Then we know that \(k(x)\), \(p\) and \(q\) are constant on \(W_0\) and \(\lambda(x)\) and \(\mu(x)\) are differentiable functions. Then, since \((p - 1)\lambda + (q - 1)\mu = 0\) holds, Lemmas 1–7 are valid. Moreover, by a similar argument to Proposition 2.1 of [3] we know that (2.5) cannot occur.

In fact, by Lemmas 6 and 7, we know \(p = q (=: p_0) < n - p - q\). Let \(\mathcal{C} = \mathcal{C}_a\).

Since we have

\[0 = g(R(X_a, X_b) Y_i, X_a) = \sum_r g(\nabla X_a Y_j, Z_r) g(\nabla X_a Z_r, X_a),\]

\((\nabla X_a Y_i)_0\) is orthogonal to \(\mathcal{C}\), or

\[g(\nabla X_a X_a, Z_\rho) = 0, \quad \text{for } 2p_0 + 1 \leq \rho \leq 3p_0,\]

using a basis \(Z_\rho, \rho = 2p_0 + 1, \ldots, 3p_0\) of \(\mathcal{C}\). Suppose \(n - p - q > p_0\) and \(\mathcal{E}\) is the orthogonal complement of \(\mathcal{C}\) in \(T_0\) on \(G\). Let \(Z_\sigma, 3p_0 + 1 \leq \sigma \leq n, \) be a basis of \(\mathcal{E}\). Then from

\[0 = g(R(X_a, Z_\sigma) Z_\tau, Y_i) = \sum_\rho g(\nabla Z_\sigma Z_\tau, Z_\rho) g(\nabla X_a Z_\rho, Y_i),\]
for $\sigma, \tau \geq 3p_0 + 1$, we obtain

$$\tilde{\nabla}_{Z_{\sigma}} Z_{\tau} = \sum_{\omega = 3p_0 + 1}^{n} g(\nabla_{Z_{\sigma}} Z_{\tau}, Z_{\omega}) Z_{\omega},$$

where $\tilde{\nabla}$ denotes the covariant differentiation for the Riemannian connection on $E^{n+1}$. Thus $\mathcal{E}$ is an involutive distribution on $G$ whose leaf is totally geodesic. For $Z_{\sigma} \in \mathcal{E}$, let $L(s)$ be the geodesic whose tangent vector is $Z_{\sigma}$. Note that $L(s)$ can be extended completely even if $g(\nabla_{Y_{\sigma}} Z_{\tau}, X_{\sigma}) = 0$ at some point on it, since a leaf of $T_0$ is complete. Moreover

$$(1.5)'' \quad Z_{\sigma} g(\nabla_{X_{\sigma}} X_{\sigma}, Z_{\sigma}) = g(\nabla_{X_{\sigma}} X_{\sigma}, Z_{\sigma})^2$$

holds for all $s \in E^1$ by (1.5). Then contradiction is shown in the same way as the proof of Lemma 6. Therefore we conclude $n - p - q = p_0$ and $g(\nabla_{X_{\sigma}} X_{\sigma}, Z_{\tau}) = 0$ for all $r$. Thus, by means of Lemma 5, (2.5) cannot occur.

Hence either (2.3) or (2.4) holds on $W_0$. If (2.4) holds on $W_0$, then the same argument as $\{5\}$ shows $\lambda$ is a constant function on $W_0$. Now assume (2.3) (resp. (2.4)) holds on $W_0$. We show that $W_0$ is actually equal to $M$. Suppose $W_0 \neq M$ and let $x$ be a point of $W_0 - W_0$. By the continuity argument for the characteristic polynomial of $A$, we see that (2.3) (resp. (2.4)) holds at $x$. Thus $W_0$ is open and closed so that $W_0 = M$ and thus (2.1) is satisfied on $M$.

REFERENCES


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