A CORRECTION NOTE ON "GENERALIZED HEWITT-SAVAGE THEOREMS FOR STRICTLY STATIONARY PROCESSES"

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Abstract. Conditions on the distribution of a process \( \{X_n, n \in I\} \) are given under which the invariant, tail and exchangeable \( \sigma \)-fields coincide; the index set \( I \) is either the positive integers or all the integers. The results proven here correct similar statements given in [3].

1. Let \( \{X_n, n \in I\} \) be a sequence of real-valued r.v.'s on the probability space \((\mathbb{R}^\infty, \mathcal{B}^\infty, \mathbb{P})\), let \( \mathcal{I}, \mathcal{T}, \) and \( \mathcal{E} \) be the invariant, tail, and exchangeable \( \sigma \)-fields (see [3] for definitions and terminology), and consider the case where \( I \) is the set of positive integers \( J \).

It is well known (see [2, p. 39; or 4]) that without reference to the probability \( \mathbb{P} \), the following strict inclusions always hold:

\[
1 \quad \mathcal{I} \subset \mathcal{T} \subset \mathcal{E}.
\]

Hence, for any probability \( \mathbb{P} \):

\[
2 \quad \mathcal{I} \subset \mathcal{T} \subset \mathcal{E}(\mathbb{P}).
\]

Looking at (1) and (2) one can see that Theorem 1 in [3] is erroneous. The inaccuracies in [3] stem from not considering separately the case where \( I \) is \( J \), the positive integers, and the case where \( I \) is \( Z \), the integers.

2. \( Z \) setup. In this case one can define \( \mathcal{I} \) and \( \mathcal{E} \) as before mutatis mutandis (now \( T \) is onto as well as 1-1, and the permutations move around a finite number of possibly negative and positive coordinates); there are, however, several \( \sigma \)-fields that could merit being called "tail \( \sigma \)-field". (For a discussion of these \( \sigma \)-fields, and many more things related to this note and to [1], see [4].) We will be satisfied here considering \( \mathcal{T} \) to be \( \bigcap_{n=1}^{\infty} \sigma(X_i, |i| \geq n) \), where \( \sigma(X_i, i \in I) \) denotes the \( \sigma \)-field generated by the variables \( X_i, i \in I \).

In this setup it is known that

\[
3 \quad \mathcal{T} \subset \mathcal{E}.
\]

The inclusion is strict and no other inclusion is valid among \( \mathcal{I}, \mathcal{T}, \mathcal{E} \) in this setup (see [4]). From (3) it is obvious that for any probability \( \mathbb{P} \):

\[
4 \quad \mathcal{T} \subset \mathcal{E}(\mathbb{P}).
\]
3. Now we will give conditions under which the inclusions (2) and (4) can be reversed.

Let $T_n \in \Sigma$ be defined in the $J$ setup by: $T_n \omega_k = (\omega)_k$ for $k \geq n + 1; T_n \omega_k = (\omega)_{k+1}$ for $k < n$. And in the $Z$ setup by: $T_n \omega_k = (\omega)_k$ for $|k| > n + 1; T_n \omega_k = (\omega)_{k+1}$ for $|k| < n$. It is easily seen that $T_n^{-1} C = T^{-1} C$ for every cylinder $C \in \sigma(X_1, \ldots, X_{n-1})$ in the $J$ setup and for every cylinder $C \in \sigma(X_i, |i| < n - 1)$ in the $Z$ setup.

Let $P \circ T^{-n}$, $P_n$ be the measures on $\mathfrak{B}^\infty$ defined by $(P \circ T^{-n}) A = P(T^{-n} A)$ and $P_n(A) = P(T_n^{-1} A)$ for $n = 1, 2, \ldots$.

Let $\ll$ denote absolutely continuity of measures.

**Theorem 1.** In the $J$ setup, if $P \circ T^{-1} \ll P$ and $P_n \ll P$ uniformly in $n$, then $\mathcal{S} = \mathcal{S}(P)$.

**Proof.** It is enough to prove $\mathcal{S} \subset \mathcal{S}(P)$. Let $A \in \mathcal{S}$ and let $C$ be a cylinder in $\sigma(X_1, \ldots, X_{n-1})$ for $n$ to be determined later. We have

\[
P(A \Delta T^{-1} A) = P(A \Delta T_n^{-1} C) + P(T_n^{-1} C T^{-1} A) = P(T_n^{-1} A \Delta T^{-1} A) + P(T^{-1} C \Delta T^{-1} A)
\]

Let $\varepsilon > 0$ be arbitrary. Find $\delta$ (independent of $n$) such that $P(A) < \delta$ implies $P(T_n^{-1} A) < \varepsilon/2$ and $P(T^{-1} A) < \varepsilon/2$.

Then $P(A \Delta T^{-1} A) < \varepsilon$. Hence $P(A \Delta T^{-1} A) = 0$, i.e., $A = T^{-1} A$.

**Theorem 2.** In the $Z$ setup, if $P \circ T^{-n} \ll P$ and $P_n \ll P$ uniformly in $n$, then $\mathcal{S} = \mathcal{S}(P)$.

**Proof.** It suffices to prove (i) $\mathcal{S} \subset \mathcal{S}(P)$ and (ii) $\mathcal{S} \subset \mathcal{S}(P)$. The proof of (i) is the same as in Theorem 1 mutatis mutandis. For (ii), let $A \in \mathcal{S}$ and $\varepsilon > 0$ be arbitrary. Find $\delta$ such that $P(T^{-n} B) < \delta$ for all $n$ whenever $P(B) < \delta$ and a cylinder $C \in \sigma(X_i, |i| < m)$ such that $P(A \Delta C) < \delta$. Then $P(A \Delta T^{-m} C) + P(T^{-m} (A \Delta C)) < \varepsilon$ and hence $P(A \Delta T^{-m} C) = 0$. Consider $D = T^{-m} C$. $T^{-n} D \in \sigma(X_i, |i| \geq n)$. Take $E = \lim \sup T^{-n} D$. Then $E \in \mathcal{S}$ and $P(A \Delta E) = 0$. This finishes the proof.

4. In proving Theorems 1 and 2 we have not used the assumption in [3]:

\[
(5) \quad \text{for each } \sigma \in \Sigma, \ P(\sigma^{-1} A) = 0 \text{ when } P(A) = 0.
\]

An example is given there, where supposedly

\[
(6) \quad \mathcal{S} \subset \mathcal{S} \subset \mathcal{S}(P) \text{ but } \mathcal{S} = \mathcal{S} \text{ does not hold}
\]

because (5) is not fulfilled.

The example is the following: consider the probability measure $P$ determined by assigning probability $1/2$ to each of the sequences $(1,0,1,0,\ldots)$ and $(0,1,0,1,\ldots)$. To see that (6) is incorrect, think of $P$ as a two-state homogeneous Markov chain with (stationary) initial distribution $\pi(0) = \pi(1) = 1/2$, and transition probabilities $p_{00} = p_{11} = 0$, $p_{10} = p_{01} = 1$. Clearly this chain has one ergodic class $\{0,1\}$ and two periodic classes $\{0\}$ and $\{1\}$ of states.

In [1], Blackwell and Freedman (see also Freedman [2]) characterize $\mathcal{S}, \mathcal{F}$ and $\mathcal{S}$ when $X_n$ is a homogeneous recurrent countable Markov chain. Applying those
results in our case (regardless of the value of $\pi(0)$ and $\pi(1)$ insofar as $0 < \pi(0) < 1$) it is plain to see that $\mathcal{F} = \text{trivial}(P)$, whereas $\mathcal{F} = \mathfrak{g}(P)$ is the $\sigma$-field generated by the two one-point atoms $\{(1,0,1,0,\ldots)\}$ and $\{(0,1,0,1,\ldots)\}$, so $\mathcal{F} \subset \mathcal{F} = \mathfrak{g}(P)$ and (6) is invalid.

Note that this Markov chain, though strictly stationary, does not satisfy the hypothesis $P_{n} \ll P$ required in Theorem 1 of [3] because the set $\{\omega\} = \{(1,0,1,0,\ldots,1,0,0,1,0,1,0,\ldots)\}$ (where _ denotes the $n$th position) has $P$-measure 0, but since $T_{n}^{-1}\omega = (0,1,0,1,0,\ldots)$, $\{\omega\}$ has $P \circ T_{n}^{-1}$-measure $1/2$.

5. Using this characterization of $\mathcal{F}$, $\mathcal{F}$, $\mathcal{E}$ for the Markov chain case, we can detect an error in the proof of Theorem 2 in [3], where it is claimed that if $f$ is the indicator of an $\mathcal{E}$-set, then $Tf$ is also in $\mathcal{E}$, i.e., if $A$ is exchangeable, $T^{-1}A$ is exchangeable. To see that this is not the case, even modulo $P$, where $P$ is a probability under which $X_{n}$ is strictly stationary, consider the example of [2, p. 46]: a Markov chain $\{X_{n}, n \geq 1\}$ with three states, whose nonzero transition probabilities are $p_{12} = p_{23} = 1$, $p_{31} = p_{32} = 1/2$. $\mathcal{E}$ is nontrivial, in fact its $P$-atoms are $\{X_{1} = 3\}$ and $\{X_{1} \in \{1,2\}\}$, and $T^{-1}\{X_{1} = 3\} = \{X_{1} = 2\}(P)$, and this latter set does not belong to $\mathcal{E}$.

REFERENCES


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