ON THE AUTOMORPHISM GROUP
OF A LINEAR ALGEBRAIC MONOID

MOHAN S. PUTCHA

Abstract. Let $S$ be a connected regular monoid with zero. It is shown that an automorphism of $S$ is inner if and only if it sends each idempotent of $S$ to a conjugate idempotent. In the language of semigroup theory, the automorphism group of $S$ maps homomorphically into the automorphism group of the finite lattice of $\mathcal{J}$-classes of $S$, and the kernel of this homomorphism is the group of inner automorphisms of $S$. In particular, if the $\mathcal{J}$-classes of $S$ are linearly ordered, then every automorphism of $S$ is inner.

Throughout this paper $Z^+$ will denote the set of all positive integers and $K$ an algebraically closed field. $\mathfrak{M}_n(K)$ denotes the monoid of all $n \times n$ matrices over $K$. $\text{GL}(n, K)$ denotes the group of units of $\mathfrak{M}_n(K)$, and $\text{SL}(n, K)$ the group of matrices of determinant $1$ in $\mathfrak{M}_n(K)$. We will follow the notation and terminology of [2,4] concerning linear algebraic monoids. Let $S$ be a connected algebraic monoid with group of units $G$. By an automorphism of $S$ is meant a semigroup automorphism $\sigma$ of $S$ such that both $\sigma$ and $\sigma^{-1}$ are polynomial maps. An automorphism $\sigma$ of $S$ is inner if there exists $g \in G$ such that $\sigma(a) = g^{-1}ag$ for all $a \in S$. We let $\mathcal{R}(S)$ denote the finite lattice of all regular $\mathcal{J}$-classes of $S$, and $E(S)$ the partially ordered set of all idempotents of $S$. It follows from the work of the author [4,5] and Renner [7,8] that $S$ is regular if and only if the closure of the radical of $G$ is a Clifford semigroup. In particular, if $S$ has a zero then $S$ is regular if and only if $G$ is a reductive group.

Theorem 1. Let $S$ be a connected regular monoid with zero and $\sigma$ an automorphism of $S$. Then $\sigma$ is an inner automorphism of $S$ if and only if $\sigma(J) = J$ for all $J \in \mathcal{R}(S)$ (i.e. $\sigma(e)$ is a conjugate of $e$ for all $e \in E(S)$).

Proof. Suppose $\sigma(J)$ for all $J \in \mathcal{R}(S)$. We must show that $\sigma$ is inner. Let $G$ denote the group of units of $S$, and let $T$ be a maximal torus of $G$. Suppose first that $T = G$. Then $\sigma(e) = e$ for all $e \in E(T)$. We prove by induction on $\dim T$ that $\sigma$ is the identity map. First suppose that $\dim T = 1$. Then by [1, Exercise 4, p. 57] either $\sigma(t) = t$ for all $t \in T$, or else $\sigma(t) = t^{-1}$ for all $t \in T$. In the latter case $t\sigma(t) = 1$ for all $t \in T$ and, hence, for all $t \in \overline{T}$. Since $0 \in \overline{T}$, this is a contradiction. So let $\dim T > 1$. Let $F = \{t \in T \mid \sigma(t) = t\}^c$. Let $e \in E(T)$, $e \neq 0$. Let $T_e = \{a \in T \mid ae = e\}^c$. 
Since $\sigma(e) = e$, $\sigma(T_e) = T_e$. Since $e$ is the zero of $\overline{T}_e$, we see by the induction hypothesis that $T_e \subseteq F$. Thus $e \in \overline{F}$. So $E(\overline{T}) \setminus \{0\} \subseteq \overline{F}$. There exists $f \in E(\overline{T})$, $f \neq 1, 0$. So \cite[Theorem 1.4]{2} there exists $h \in E(\overline{T})$, $h \neq 0$, such that $fh = 0$. Since $f$, $h \in \overline{F}$, $0 \in \overline{F}$. So $E(\overline{T}) = E(\overline{F})$. By \cite[Theorem 1.4]{2}, $\dim T = \dim F$. Thus $T = F$ and $\sigma$ is the identity map. In particular, the automorphism group of $\overline{T}$ is isomorphic to a subgroup of the group of automorphisms of $E(\overline{T})$ and, hence, is a finite group.

Let us now consider the general case. Since all maximal tori of $G$ are conjugate, we can assume without loss of generality that $\sigma(T) = T$. Let $\Lambda \subseteq E(\overline{T})$ be a cross-section lattice of $S$ (see \cite{6}). In other words, if $e, f \in \Lambda$, $f \in S \Lambda S$, then $e \geq f$, each idempotent of $S$ is conjugate to an idempotent in $\Lambda$, and no two idempotents in $\Lambda$ are conjugate. So $\sigma(\Lambda) \subseteq E(\overline{T})$ is also a cross-section lattice of $S$. By \cite[Theorem 11]{6} there exists $u \in W$, the Weyl group of $G$ relative to $T$, such that $\sigma(\Lambda) = u^{-1}\Lambda u$. Thus, without loss of generality, we can assume that $\sigma(\Lambda) = \Lambda$. Let $e \in \Lambda$. Then $e, \sigma(e) \in \Lambda$. By hypothesis, $e \nmid \sigma(e)$. Since $\Lambda$ is a cross-section lattice of $S$, $e = \sigma(e)$. So $\sigma(e) = e$ for all $e \in \Lambda$. Let $F = \{t \in T | \sigma(t) = t\}^c$. Since the automorphism group of $\overline{T}$ is finite, we see that there exists $k \in Z^+$ such that $\sigma^k(a) = a$ for all $a \in \overline{T}$. So by \cite[Lemma 1.13]{3}, $\Lambda \subseteq \overline{F}$. Since $\Lambda$ contains a maximal chain of $E(\overline{T})$, we see by \cite[Theorem 1.4]{2} that $\dim F = \dim T$, so $T = F$. Thus $\sigma(t) = t$ for all $t \in T$. Let $\Gamma$ be a maximal chain in $E(\overline{T})$, and let $B = \{a \in G | ae = eae \text{ for all } e \in \Gamma\}$. Since $\sigma(\Gamma) = \Gamma$, we see that $\sigma(B) = B$. Since $G$ is a reductive group, we see by \cite[Theorem 4.5]{4} that $B$ is a Borel subgroup of $G$. Clearly $T \subseteq B$. We see by \cite[Theorem 27.4(b) and 9, Theorem 11.4.3]{1} that $\sigma$ is an inner automorphism.

Example. Let $S$ denote the Zariski closure in $\mathfrak{h}_3(K) \times \mathfrak{h}_3(K)$ of \{$(aA, a^{-1}_0A) \mid a \in K, A \in \text{SL}(3, K)$\}. Then $S$ is a connected regular monoid with zero, $\mathfrak{h}(S) = \{G, J, J_1, J_2, J_3\}$ with $G > J > J_1 > J_2 > 0$, $i = 1, 2$. Let $e, e_1, e_2$, denote diagonal matrices with the respective diagonals being \{(1, 0, 0), (0, 0, 1)), ((1, 0, 0), (0, 0, 0)), ((1, 0, 0), (0, 0, 0)), ((0, 0, 0), (0, 0, 1)). Then $e \in J$, $e_1 \in J_1$, $e_2 \in J_2$ and \{1, $e, e_1, e_2, 0\}$ is a cross-section lattice of $S$. Let $\sigma: S \to S$ be given by $\sigma(A, B) = (B, A)$. Then $\sigma$ is an automorphism of $S$ which is not inner. Note that $\sigma$ induces a nontrivial automorphism of $\mathfrak{h}(S)$: $\sigma(G) = G$, $\sigma(J) = J$, $\sigma(0) = 0$, $\sigma(J_1) = J_2$, $\sigma(J_2) = J_1$.

Theorem 1 can be restated as follows.

**Theorem 2.** Let $S$ be a connected regular monoid with zero. Then the automorphism group of $S$ maps homomorphically into the automorphism group of $\mathfrak{h}(S)$. The kernel of this homomorphism is the group of inner automorphisms of $S$.

**Corollary.** Let $S$ be a connected regular monoid with zero such that the $\mathfrak{q}$-classes of $S$ are linearly ordered. Then every automorphism of $S$ is inner.

**Remark.** The above corollary applies to the multiplicative monoid $\mathfrak{h}_n(K)$. Note that the map $A \to (A^{-1})^T$ is an automorphism of $\text{GL}(n, K)$ which is not inner.

**References**

5. ____, Reductive groups and regular semigroups, J. Algebra (submitted).
6. ____, Idempotent cross-sections of $\mathfrak{f}$-classes, Semigroup Forum (to appear).
8. ____, Reductive monoids are von-Neumann regular (to appear).

DEPARTMENT OF MATHEMATICS, NORTH CAROLINA STATE UNIVERSITY, RALEIGH, NORTH CAROLINA 27650