AN ERGODIC THEOREM FOR SEMIGROUPS OF CONTRACTIONS

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Abstract. An ergodic theorem for semigroups of nonlinear contractions having precompact trajectories in a Banach space is proved.

1. The main result. Throughout this note X will be a real Banach space, C ⊂ X a closed subset of X and S(t), t ≥ 0, a semigroup of contractions on C, that is a family of mappings S(t): C → C, t ≥ 0, satisfying:
   (i) \( \lim_{t \to 0} S(t)x = S(t_0)x \) for \( t_0 \geq 0, \ x \in C \).
   (ii) \( S(t + s)x = S(t)S(s)x \) for \( t, s \geq 0, \ x \in C \).
   (iii) \( ||S(t)x - S(t)y|| \leq ||x - y|| \) for \( t \geq 0, \ x, y \in C \).

For \( x \in C \) we denote by \( \omega(x) = \{ S(t)x : t \geq 0 \} \) the trajectory starting at \( x \) and by
\[
\omega(x) = \left\{ y : y = \lim_{n \to \infty} S(t_n)x, \text{ for some sequence } t_n \to \infty \right\},
\]
the possibly empty \( \omega \)-limit set of \( X \). If \( \omega(x) \neq \emptyset \) then it follows from its definition that \( \omega(x) \) is invariant under \( S(t), \ t \geq 0 \), i.e. \( S(t): \omega(x) \to \omega(x) \) for \( t \geq 0 \) and

\[
\lim_{t \to \infty} \text{dist}(S(t)x, \omega(x)) = 0,
\]
where dist(\( z, B \)) is the distance between the point \( z \) and the set \( B \). Assuming, as we will do below, that for some \( x \in C \) the trajectory \( \omega(x) \) is precompact, it follows easily that \( \omega(x) \) is nonempty and compact. In this case, \( \omega(x) \) can be given the structure of a compact commutative group and the following much stronger assertion, which is our main result, holds.

Theorem 1 (the ergodic theorem). Let \( X, Y \) be real Banach spaces, \( C \subset X \) be closed and let \( S(t), t \geq 0, \) be a semigroup of contractions on \( C \). If for some \( x \in C \) the trajectory \( \gamma(x) \) is precompact, then \( \omega(x) \) is a compact commutative group, and for every \( f: C \to Y \) which is uniformly on bounded subsets of \( C \) we have

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T f(S(t)x) \, dt = \int_{\omega(x)} f(\xi) \, d\xi,
\]
where \( d\xi \) is the unique normalized Haar's measure on \( \omega(x) \).

2. The proof of Theorem 1. Let \( C \subset X \) be a closed subset of the Banach space \( X \) and let \( S(t), t \geq 0, \) be a semigroup of contractions on \( C \). A subset \( \Omega \) of \( C \) is called minimal under \( S(t), t \geq 0, \) if it is the closure of the trajectory \( \gamma(y) = \{ S(t)y : t \geq 0 \} \)

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for every $y \in \Omega$; it is strongly invariant under $S(t)$, $t \geq 0$, if for every $t \geq 0$, $S(t)$ is a homeomorphism of $\Omega$ onto itself so that $S(t)$, $t \geq 0$, can be extended as a continuous group on $\Omega$. The set $\Omega$ is equi-almost periodic under $S(t)$ if it is strongly invariant and for every $\varepsilon > 0$ the set of real numbers with the property $\sup_{y \in \Omega} \|S(t)y - y\| < \varepsilon$ is relatively dense. The following proposition, whose proof can be found for example in [4, Theorem 1], is a standard result from the theory of dynamical systems.

**Proposition 2.** If for some $x \in C$, $\omega(x) \neq \emptyset$, then $\omega(x)$ is minimal and strongly invariant under $S(t)$. For each $t \in \mathbb{R}$, $S(t)$ is an isometry on $\omega(x)$. Moreover, if $\omega(x)$ is compact, then it is equi-almost periodic under $S(t)$.

The proof of Theorem 1 will follow easily from the next two lemmas.

**Lemma 3.** Let $F: C \to Y$ be uniformly continuous on bounded subsets of $C$. If the trajectory $\gamma(x)$ is bounded and $\omega(x) \neq \emptyset$ then

$$\lim_{T \to \infty} \left\| \frac{1}{T} \int_0^T f(S(t)x) \, dt - \frac{1}{T} \int_0^T f(S(t)y) \, dt \right\| = 0$$

for every $y \in \omega(x)$.

The proof of this simple lemma is omitted. It can be found, e.g., in [6, Proposition 4.1].

If $\omega(x)$ is a nonempty compact subset of $C$ there is a standard way to endow it with a commutative groups structure (see e.g. [5, Theorem 8.16, p. 394]), and hence there is a unique normalized Haar measure on it which will be denoted by $d\xi$. Moreover, if $\omega(x) \neq \emptyset$ is compact it follows from Proposition 2 that it is a minimal set consisting of almost periodic motions, and hence by [5, Theorem 9.34, p. 510] we have

**Lemma 4.** Let $\omega(x) \neq \emptyset$ be compact and $y \in \omega(x)$. For every continuous real valued function $h: \omega(x) \to \mathbb{R}$ we have

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T h(S(t)y) \, dt = \int_{\omega(x)} h(\xi) \, d\xi.$$

We turn now to the proof of Theorem 1.

**Proof of Theorem 1.** From the precompactness of the trajectory $\gamma(x)$ it follows that it suffices to prove the theorem only for functions $f \in C(\omega(x); Y)$ (the space of continuous $Y$-valued functions on $\omega(x)$). Since $\omega(x)$ is compact each function $f \in C(\omega(x); Y)$ can be uniformly approximated by functions $g_n(z)$ of the form $g_n(z) = \sum_{k=1}^n h_k(z)e_k$ where $h_k(z): \omega(x) \to \mathbb{R}$ is continuous and $e_k \in Y$ for $1 \leq k \leq n$. From Lemma 4 it follows readily that the theorem is true for functions $g_n$ of this form, and therefore by the uniform continuity of $f \in C(\omega(x); Y)$ and the functions $g_n \in C(\omega(x); Y)$, it is also true for any $f \in C(\omega(x); Y)$ and the proof is complete.

**3. Concluding remarks.** It is well known that if $A$ is an $m$-accretive operator in a Banach space $X$ (for the definitions and properties of such operators see e.g. [1 and 3]) then it generates a semigroup of contractions $S(t)$, $t \geq 0$, on $D(A)$ given by the exponential formula

$$S(t)x = \lim_{n \to \infty} \left( I + \frac{t}{n} A \right)^n x \quad \text{for } x \in D(A).$$

For the proof of (5) see [3].
The main assumption of the ergodic theorem is the precompactness of the trajectory \( \gamma(x) \) for some \( x \in C \). This condition is clearly satisfied for all \( x \in C \) if the semigroup \( S(t), t \geq 0 \), is compact for \( t \geq 0 \), i.e. for every \( t > 0 \), \( S(t) \) is a compact operator. A characterization of such compact semigroups, in terms of their \( m \)-accretive generator, is given in [2].

The compactness of the semigroup \( S(t), t \geq 0 \), is of course not necessary for the precompactness of all the trajectories of \( S(t), t \geq 0 \). It is shown in [4, Theorem 3] that if \( A \) is \( m \)-accretive, \( 0 \) is in the range of \( A \) and the everywhere defined contractions \( (I + tA)^{-1} \) are compact for all \( t > 0 \), then all the trajectories \( S(t), t \geq 0 \), are precompact and thus one can apply Theorem 1 to such semigroups.

Finally we note that Theorem 1 is an extension of a similar result in [6, Theorem 4.5] which deals with the special case where \( X \) is a real Hilbert space. The conditions there assure that \( \omega(x) \) lies in a finite-dimensional subspace of \( X \) and it is nonempty and bounded. Hence \( \omega(x) \) is clearly compact and the situation is similar to that of Theorem 1.

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References


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