ZERO-FREE PARABOLIC REGIONS
FOR POLYNOMIALS WITH COMPLEX COEFFICIENTS

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Abstract. Recent results by P. Henrici, E. B. Saff and R. S. Varga on zero-free parabolic regions for sequences of polynomials generated from three-term recurrence relations with real coefficients are generalized to complex coefficients by continued fraction methods. Especially, it is shown that all zeros of the generalized Bessel polynomials $Y_{m}^{(\delta)}(z)$ for complex $\delta$ are contained in a cardioid region, which generalizes a result of E. B. Saff and R. S. Varga for real $\delta$.

1. Introduction. In [6] E. B. Saff and R. S. Varga showed that certain sequences of polynomials defined by three-term recurrence relations with real coefficients have no zeros in a parabolic region. From this it is deduced (cf. [2, pp. 75–89]) that all zeros of the $m$th generalized Bessel polynomial $Y_{m}^{(\delta)}(z)$, depending on the parameter $\delta$, are contained in the cardioid region

$$\{z = re^{i\theta} \in \mathbb{C}: 0 < r < (1 + \cos \theta)/(m + 1 + \delta), |\theta| < \pi \} \cup \{2/(m + 1 + \delta)\},$$

provided $m + 1 + \delta > 0$. In [3] P. Henrici generalized the results of [6] so as to apply to interpolation polynomials with real interpolation points. In the present paper the continued fraction method of P. Henrici [3] is generalized so as to yield zero-free parabolic regions also for polynomials satisfying three-term recursion relations with complex coefficients, which include interpolation polynomials with complex interpolation points. As an interesting application, it is shown that all zeros of $Y_{m}^{(\delta)}(z)$, $m \geq 2$, are contained in the open cardioid region

$$\{z = re^{i\theta} \in \mathbb{C}: 0 < r < (1 + \cos \theta)/(m + 1 + \delta \cos^2(\chi/2), |\theta| < \pi \},$$

where $m + 1 + \delta = |m + 1 + \delta| e^{i\chi} \neq 0$ and $|\chi| < \pi$, i.e. $\delta$ is complex, provided $m + 1 + \delta$ is not $\leq 0$. In particular, for $0 < |\chi| < \pi/2$ this region is properly contained in the open cardioid region

$$\{z = re^{i\theta} \in \mathbb{C}: 0 < r < (1 + \cos \theta)/(m + 1 + \text{Re} \delta), |\theta| < \pi \}.$$
in [3] we observe that \( q_n \) is the denominator of the continued fraction

\[
    \frac{1}{b_1 + \frac{z_2 a_2}{b_2 + \frac{z_3 a_3}{b_3 + \cdots + \frac{z_{n+1} a_{n+1}}{b_{n+1}}}}}.
\]

Since \( p_n, q_n \) cannot vanish simultaneously if all \( a_n z_n \neq 0 \), we first assume \( z_n \neq 0 \) for \( n \geq 2 \). Later on we will see that all results below remain valid if some \( z_n = 0 \). It then suffices to show that \( w_n \neq \infty \) in order to obtain \( q_n \neq 0 \). (2) can be written as

\[
    w_n = s_1 \circ \cdots \circ s_n(z_{n+1}),
\]

where \( \mathbb{C} \) is the closed complex plane

\[
    s_n(u) := z_n \left( 1 - \frac{a_n}{b_n + u} \right), \quad n \geq 2,
\]

and

\[
    s_1(u) := \frac{1}{b_1 + u}, \quad u \in \mathbb{C}.
\]

We want to determine closed half-planes \( H_n \subset \mathbb{C}, n \in \mathbb{N} \), and (large) sets \( \mathcal{P}_n \subset \mathbb{C}, n \geq 2 \), such that for \( z_n \in \mathcal{P}_n \),

\[
    H_n := \left\{ \xi \in \mathbb{C} : \text{Re} \left( \sum \frac{\ell_n - z_n}{z_n c_n} \right) > 0 \right\},
\]

where \( \ell_n \in \mathbb{R}, d_n \in \mathbb{C}; \text{Re} e^{i\varphi_n d_n} > 0 \).

Since \( s_n(-b_n) = \infty \), \( D_n \) is a finite closed disk iff \( -b_n \notin H_n \) or \( \text{Re} e^{i\varphi_n (b_n - d_n)} > 0 \), which we want to assume for \( n \geq 2 \). Next, we evaluate \( D_n \) explicitly. Let \( b_n^* \) be symmetric to \( -b_n \) with respect to \( \partial H_n \), the boundary of \( H_n \). Then

\[
    b_n^* = -d_n + e^{-2i\varphi_n (b_n - d_n)},
\]

and the center of \( D_n \) is given by

\[
    \xi_n := s_n(b_n^*) = z_n(1 - c_n),
\]

where

\[
    c_n := \frac{a_n}{(b_n + b_n^*)} = e^{i\varphi_n a_n/2} \text{Re} e^{i\varphi_n (b_n - d_n)}, \quad n \geq 2.
\]

Since \( z_n = s_n(\infty) \in \partial D_n \), the boundary of \( D_n \), the radius of \( D_n \) is given by

\[
    r_n := \left| z_n - \xi_n \right| = \left| z_n c_n \right|.
\]

Hence

\[
    D_n = \left\{ \xi \in \mathbb{C} : \left| \xi - z_n + z_n c_n \right| < \left| z_n c_n \right| \right\}, \quad n \geq 2.
\]

For \( n \geq 2 \), \( D_n \subset H_{n-1} \) is equivalent to \( \text{Re} e^{i\varphi_n (\xi_n + d_{n-1})} - r_n \geq 0 \) or, explicitly,

\[
    \text{Re} e^{i\varphi_n (z_n - z_n c_n + d_{n-1})} - \left| z_n c_n \right| \geq 0,
\]

or

\[
    \text{Re} e^{i\varphi_n (z_n + d_{n-1})} \geq \text{Re} \left( e^{i\varphi_n (z_n c_n)} + \left| z_n c_n \right| \right), \quad n \geq 2.
\]

Since here the right side is \( \geq 0 \), \( z_n \in H_{n-1} \) follows and, hence, \( P_n \subset H_{n-1} \) must hold for \( n \geq 2 \). We now define \( P_n \) by

\[
    P_n := \left\{ \xi \in \mathbb{C} : \text{Re} \left( e^{i\varphi_n (\xi c_n)} + \left| \xi c_n \right| \leq 2 h(c_n) \text{Re} e^{i\varphi_n (\xi + d_{n-1})} \right) \right\},
\]

where the real-valued function \( h \) satisfies \( 0 < h(c_n) \leq 1/2 \), \( n \geq 2 \).

Obviously (8) holds for each \( z_n \in P_n \), and different choices of the function \( h \) lead to different conditions on \( a_n, b_n, z_n \). Next, \( a_n \neq 0 \) implies \( c_n \neq 0 \). Hence (9) is
equivalent to
\[
(10) \quad P_n = \{ \xi \in \mathbb{C} : |\xi| \leq \text{Re}(e^{i\varphi_n - \epsilon}(2h(c_n) - c_n) / |c_n|) + 2(h(c_n) / |c_n|) \text{Re}(e^{i\varphi_n - \epsilon}d_n) \}
\]
where \( P_n \subset H_{n-1}, \ n \geq 2 \). Observe that each \( P_n \) is convex. Finally, assume that \( z_1, \ldots, z_m \neq 0, \ z_{m+1} = \cdots = z_{m+j} = 0, \ z_{m+j+1} \neq 0 \) for some \( m, j \geq 1 \). Then
\[
q_1, \ldots, q_m \neq 0
\]
by the preceding considerations and, by (1),
\[
q_{m+v} = (b_{m+v} + z_{m+v+1})q_{m+v-1}, \quad 1 \leq v \leq j,
\]
which is \( \neq 0 \) because \( z_{m+v+1} \in P_{m+v+1} \subset H_{m+v} \) and \(-b_{m+v} \not\in H_{m+v}\). For \( n \in \mathbb{N} \) then
\[
q_{m+n-1} = q_{m+n-1}q_n^* \text{ holds, where } q_n^* \text{ satisfies}
\]
\[
q_n^* = (b_{(m+j-1)+n} + z_{(m+j-1)+n+1})q_n^* - a_{(m+j-1)+n}z_{(m+j-1)+n}q_n^2
\]
with \( q_1^* = 0, q_0^* = 1 \). To these \( q_n^* \) the above considerations can be applied until, again for some \( n, z_{(m+j-1)+n} \neq 0 \) holds. But then the preceding argument can be repeated.

3. General results. In the preceding considerations we have proved

**Theorem 1.** If (cf. (1), (5), (9)) \( \text{Re}(e^{i\varphi_n}(b_n - d_n) > 0, 1 \leq n \leq N \) and \( z_n \in P_n, 2 \leq n \leq N + 1 \), or if, more generally, \( |b_1 + z_2 - z_2c_2| > |z_2c_2| \) (i.e. \(-b_1 \not\in D_2\)), \( \text{Re}(e^{i\varphi_n}(b_n - d_n) > 0, 2 \leq n \leq N \) and \( z_n \in P_n, 3 \leq n \leq N + 1 \), then \( q_n \neq 0 \) for \( 1 \leq n \leq N \). If the above conditions hold for arbitrary \( N \geq 2 \), then \( q_n \neq 0 \) for \( n \geq 1 \).

Next, we assume that \( z_{n+1} \in \partial H_n \cap \partial P_{n+1} \) for some \( n \in \mathbb{N} \). Then (9) implies
\[
\text{Re}(e^{i\varphi_n}z_{n+1}c_{n+1} + |c_{n+1}z_{n+1}|) = 0.
\]
Hence, \( \text{Im}(e^{i\varphi_n}z_{n+1}c_{n+1} = 0 \) and, from (6), we obtain
\[
(11) \quad z_{n+1} = -e^{-i(\varphi_n + \varphi_n + \psi_{n+1})} |z_{n+1}|.
\]
This and \( z_{n+1} \in \partial H_n \) imply
\[
(12) \quad \text{Re}(e^{i\varphi_n}d_n = -\text{Re}(e^{i\varphi_n}z_{n+1} = |z_{n+1}| \cos(\varphi_{n+1} + \psi_{n+1})).
\]
Hence, \( \cos(\varphi_{n+1} + \psi_{n+1}) > 0 \) since \( \text{Re}(e^{i\varphi_n}d_n > 0 \). If \( |z_{n+1}| \) from (12) is substituted into (11), we obtain
\[
(13) \quad z_{n+1} = z_{n+1} = -e^{-i(\varphi_n + \varphi_n + \psi_{n+1})} \text{Re}(e^{i\varphi_n}d_n, \quad \cos(\varphi_{n+1} + \psi_{n+1}) > 0.
\]
(\( z_{n+1} = -d_n \) if \( \varphi_n = \varphi_{n+1} = 0, a_{n+1} > 0, d_n > 0 \)).

Conversely, one verifies that \( z_{n+1} \in \partial H_n \cap \partial P_{n+1} \) holds. Therefore, \( \partial H_n \) and \( \partial P_{n+1} \) always have exactly one point, \( z_{n+1} \), in common, provided
\[
\cos(\varphi_{n+1} + \psi_{n+1}) > 0.
\]

If, in particular, \(-b_1 \not\in H_n, z_n \in P_n, v > 2, \) and \(-b_1 \in \partial H_1, \) then \( n = \infty \) or \( s_{2} \circ \cdots \circ s_{n}(z_{n+1}) = -b_1 \) can occur for some \( n \geq 2 \) only if \(-b_1 \in \partial D_2 \) also. By the above considerations this is only possible if \( z_{n+1} = z_{n+1}' \) in (13) and if all disks
\[
D_2 = s_2(H_2) \supset s_2 \circ s_3(H_3) \supset \cdots \supset s_2 \circ \cdots \circ s_{n}(H_n)
\]
touch \( \partial H_1 \) at the point \(-b_1\).
Finally, we assume that $z_n = z$, $H_n = H := \{ \xi \in \mathbb{C} : \text{Re } e^{i\varphi}(\xi + d) \geq 0 \}$, $n \geq 1$, with fixed $\varphi \in \mathbb{R}$, $d \in \mathbb{C}$ and $-b_1 \in \partial H$, $-b_1 \in \partial H$, $n \geq 2$. If for some $n \geq 2$ and $z \in \cap_{n=2}^{\infty} P_v$, $s_2 \circ \cdots \circ s_n(z) = -b_1$ holds, then $-b_1 \in \partial D_2$ also and, hence, $z = z_{n+1} \in \partial H_n$. Now $-b_1 \in \partial H \cap \partial D_2$ is equivalent to $\text{Re } e^{i\varphi}(b_1 - d) = 0$ and $|b_1 + z - z_{c_2}| = |z_{c_2}|$. If $\text{Im } e^{i\varphi}(b_1 + z - z_{c_2}) \neq 0$, then

$$|b_1 + z - z_{c_2}| > \text{Re } e^{i\varphi}(b_1 + z - z_{c_2}) = -\text{Re } e^{i\varphi}z_{c_2} = |z_{c_2}|$$

because $\text{Re } e^{i\varphi}(b_1 + z) = \text{Re } e^{i\varphi}(d + z) = 0$ and (9), since $z \in \partial H \cap \partial D_2$. But this contradicts $-b_1 \in \partial D_2$. Hence, $\text{Im } e^{i\varphi}(b_1 + z - z_{c_2}) = 0$. This yields $\text{Im } e^{i\varphi}(b_1 + z) = 0$, since $-\text{Re } e^{i\varphi}z_{c_2} = |z_{c_2}|$ implies $\text{Im } e^{i\varphi}z_{c_2} = 0$. Therefore $b_1 + z = 0$. If $n = 2$, then this, $0 = b_1 + s_2(z)$, and $z \neq 0$ imply $b_2 + z = \infty$. If $n \geq 3$, then $b_1 + z = 0$, $z \neq 0$, and $0 = b_1 + s_2 \circ \cdots \circ s_n(z)$ imply $s_3 \circ \cdots \circ s_n(z) = \infty$. But this is impossible by applying Theorem 1 to the denominators of the continued fraction

$$\frac{1}{b_3 + z - b_4 + z - \cdots},$$

since $-b_n \notin H$ for $n \geq 2$. Hence, $q_n \neq 0$ for $n \geq 2$ and $q_1 = 0$ iff $z = -b_1$. We thus have proved

**Theorem 2.** (1) If (cf. (1),(5),(9),(13)) $\text{Re } e^{i\varphi}(b_1 - d_1) = 0$ (i.e. $-b_1 \in \partial H_1$), $\text{Re } e^{i\varphi}(b_n - d_n) > 0$, $2 \leq n \leq N$ and if $z_n \in P_n$, $2 \leq n \leq N + 1$, then $q_n \neq 0$ provided $z_{n+1} \neq z_{n+1}'$ in case $|b_1 + z_2 - z_{c_2}| = |z_{c_2}|$ (i.e. $-b_1 \in \partial D_2$).

(2) If $\text{Re } e^{i\varphi}(b_1 - d_1) = 0$, $\text{Re } e^{i\varphi}(b_n - d_n) > 0$, $n \geq 2$ and $z_n \in P_n$, $n \geq 2$, then $q_n \neq 0$ for $n \geq 2$ provided $z_{n+1} \neq z_{n+1}'$, $n \geq 2$, in case $|b_1 + z_2 - z_{c_2}| = |z_{c_2}|$.

(3) Assume that all $z_n = z$ and that for fixed $\varphi \in \mathbb{R}$, $d \in \mathbb{C}$, $\text{Re } e^{i\varphi}d > 0$ holds. If $\text{Re } e^{i\varphi}(b_1 - d) \geq 0$, and $\text{Re } e^{i\varphi}(b_n - d) > 0$, $2 \leq n \leq N$, then $q_n(z) \neq 0$ for $z \in \cap_{n=2}^{N+1} P_v$ and $2 \leq n \leq N$. ($q_1(z) = 0$ iff $z = -b_1$.) If the above conditions hold for arbitrary $N \geq 2$, then $q_n(z) \neq 0$ for $z \in \cap_{v=2}^{\infty} P_v$ and $n \geq 2$.

We now choose $h(c_n) = \text{Re } c_n$ in (10). Then (10) is equivalent to

$$P_n = \{ \xi \in \mathbb{C} : |\xi| < \text{Re } (e^{i\varphi} - i\xi\text{Re } c_n/|c_n|) + 2\text{Re } (e^{i\varphi} - d_{n-1})\text{Re } c_n/|c_n| \}.$$  

By (1) and (6), $\text{Re } c_n/|c_n| = e^{-(\varphi + \psi_n)}$. Hence,

$$h(c_n) = \text{Re } c_n > 0$$

iff $\cos(\varphi_n + \psi_n) > 0$, and $c_n \leq \frac{1}{2}$ iff $\text{Re } e^{i\varphi}(b_n - a_n - d_n) \geq 0$, $n \geq 2$. If these conditions are satisfied, then $\text{Re } e^{i\varphi}(b_n - d_n) \geq \text{Re } e^{i\varphi}a_n > 0$ holds for $n \geq 2$ and, hence, Theorems 1 and 2(1)(2) yield

**Corollary 1.** Assume that $\varphi_n \in \mathbb{R}$, $d_n \in \mathbb{C}$ such that $\text{Re } e^{i\varphi_n}d_n > 0$, $n \in \mathbb{N}$. Let $P_n$, $n \geq 2$, be the parabolic region

$$P_n = \{ \xi \in \mathbb{C} : |\xi| < \text{Re } e^{i\varphi_n-\xi(\varphi_n-\psi_n)} + 2\cos(\varphi_n + \psi_n)\text{Re } e^{i\varphi_n-1}d_{n-1} \},$$

where $\cos(\varphi_n + \psi_n) > 0$, $n \geq 2$.

(1) If $\text{Re } e^{i\varphi_n}(b_1 - d_1) > 0$, $\text{Re } e^{i\varphi_n}(b_n - a_n - d_n) \geq 0$, $2 \leq n \leq N$, and $z_n \in P_n$, $2 \leq n \leq N + 1$, or if, more generally,

$$|b_1 + z_2 - z_{c_2}| > |z_{c_2}|,$$
Re $e^{i\xi}(b_n - a_n - d_n) > 0$, $2 \leq n \leq N$ and $z_n \in P_n$, $3 \leq n \leq N + 1$, then $q_n \neq 0$ for $1 \leq n \leq N$. If the above conditions hold for all $N \geq 2$, then $q_n \neq 0$ for $n \geq 1$.

(2) If $Re e^{i\xi}(b_1 - d_1) = 0$, $Re e^{i\xi}(b_n - a_n - d_n) > 0$, $2 \leq n \leq N$ and $z_n \in P_n$, $2 \leq n \leq N + 1$, then $q_n \neq 0$ provided $z_{n+1} \neq z_{n+1}'$ in case $|b_1 + z_2 - z_2 c_2| = |z_2 c_2|$. If the above conditions hold for all $N > 2$, then $q_n \neq 0$ for $n \geq 1$.

(3) If $Re e^{i\xi}(b_1 - d_1) = 0$, $Re e^{i\xi}(b_n - a_n - d_n) > 0$, $n \geq 2$, and $z_n \in P_n$, $n \geq 2$, then $q_n \neq 0$ for $n \geq 2$ provided $z_{n+1} \neq z_{n+1}'$, $n \geq 2$ in case $|b_1 + z_2 - z_2 c_2| = |z_2 c_2|$. In particular, we obtain from Theorem 2(3):

Corollary 2. Assume that in (1) $a_n = |a_n| e^{i\psi}$ and $b_n - a_n = |b_n - a_n| e^{i\phi}$ with fixed $\psi, \chi \in \mathbb{R}$ for $n \geq 2$. Assume also that $\rho := \inf_{n \geq 2} |b_n - a_n| > 0$ and put $d := \rho e^{i\chi}$. If for some $\varphi \in \mathbb{R}$, $Re e^{i\varphi}(b_1 - d_1) \geq 0$, $\cos(\varphi + \psi) > 0$ and $\cos(\varphi + \chi) > 0$ (i.e. $Re e^{i\varphi} > 0$), then the polynomials $q_n(z)$, defined by

\[ q_n(z) = (b_n + z)q_{n-1}(z) - a_n z q_{n-2}(z), \quad n \in \mathbb{N}, \quad q_{-1} := 0, \quad q_0 := 1, \]

have no zeros for $n \geq 2$ in the closed parabolic region

\[ P' = \{z \in \mathbb{C} : |z| \leq Re e^{-i\alpha} + 2\rho |\varphi + \psi| (\cos(\varphi + \psi)) \}. \]

\[ q_1(z) = 0 \iff z = -b_1. \] If $\varphi = -(\psi + \chi)/2$ and $|\chi - \psi| < \pi$, then $q_n(z)$, $n \geq 2$, have no zeros in the closed parabolic region

\[ (14) \quad P = \{z \in \mathbb{C} : |z| \leq Re e^{-i\alpha} + 2\rho \cos^2((\chi - \psi)/2)\} \supset P'. \]

Proof. Since $Re e^{i\varphi} a_n = |a_n| \cos(\varphi + \psi) > 0$ and $Re e^{i\varphi} d = \rho \cos(\varphi + \chi) > 0$, Corollary 2 follows from Theorem 2(3). Observe that

\[ 2\cos(\varphi + \psi) \cos(\varphi + \chi) = \cos(2\varphi + \psi + \chi) + \cos(\psi - \chi) \]

is maximal for $\varphi = -(\chi + \psi)/2$. Then

\[ \cos(\varphi + \psi) = \cos(\varphi + \chi) = \cos((\chi - \psi)/2) > 0 \quad \text{if} \quad |\chi - \psi| < \pi. \]

4. Application to generalized Bessel polynomials. These polynomials are defined for $n \in \mathbb{N}$ by (cf. [2, 6])

\[ Y^{(\delta)}_n(z) := 1 + \sum_{j=1}^{n} \binom{n}{j} (n + \delta + 1) \cdots (n + \delta + j)(-z/2)^j, \quad \text{where} \quad \delta \in \mathbb{C}. \]

Then for fixed $m \in \mathbb{N}$ the polynomials $q^{(m+\delta)}_n(z) := z^n Y^{(m+\delta-n)}_-(-2/z)$ satisfy

\[ q^{(m+\delta)}_n(z) = (n + m + \delta + z) q^{(m+\delta)}_{n-1}(z) - (n - 1) z q^{(m+\delta)}_{n-2}(z) \]

with $q^{(m+\delta)}_{-1} := 0, q^{(m+\delta)}_0 := 1$. Hence $q^{(m+\delta)}_n(z)$, $n \in \mathbb{N}$, is of type (1) with $b_n = n + m + \delta, a_n = n - 1$, $n \in \mathbb{N}$, and, therefore, $b_n - a_n = m + 1 + \delta$ is independent of $n$. With the notation of Corollary 2, we have, in this example,

\[ \psi = 0, \quad \rho = |m + 1 + \delta| \quad \text{and} \quad d = m + 1 + \delta = |m + 1 + \delta| e^{i\chi} = b_1. \]

If $\varphi = -\chi/2$, then (14) yields a closed parabolic region (containing $-b_1$)

\[ P = \{z \in \mathbb{C} : |z| \leq Re \xi + 2 |m + 1 + \delta| \cos^2(\chi/2)\} \]

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such that \( q_{m+S}(z) \neq 0 \) for \( n \geq 2, \ z \in P \), provided \( m + 1 + \delta \neq 0 \) and \(|\chi| < \pi\). For \( 0 < |\chi| < \pi / 2 \), \( 2 \cos^2(\chi/2) = 1 + \cos \chi > 2 \cos \chi \) holds and, therefore, \( P \) properly contains the parabolic region \( P' = \{ \xi \in \mathbb{C} : |\xi| < \Re \xi + 2(m + 1 + \Re \delta) \} \).

In particular, for \( n = m \geq 2 \), this shows that \( Y_{m+\delta}(z) \neq 0 \) for \(-2/z \in P\). Thus we have proved

**Theorem 3.** If \( m + 1 + \delta = |m + 1 + \delta| e^{i\chi} \neq 0 \) and \(|\chi| < \pi\), then all zeros of \( Y_{m+\delta}(z), \ m \geq 2, \) are contained in the open cardioid region

\[
C = \{ z = re^{i\theta} \in \mathbb{C} : 0 < r < (1 + \cos \theta) / |m + 1 + \delta| \cos^2(\chi/2), |\theta| < \pi \}.
\]

i.e. if \( m + \delta + 1 \) lies on the parabola \(|z| + \Re z = 2c > 0\), then all zeros of \( Y_{m+\delta}(z), \ m \geq 2, \) are contained in

\[
C = \{ z = re^{i\theta} \in \mathbb{C} : 0 < r < (1 + \cos \theta) / |m + 1 + \Re \delta|, |\theta| < \pi \}.
\]

\( Y_{1+\delta}(z) = 0 \) iff \( z = 2/(2 + \delta) \).

In the special case where \( m + \delta + 1 \) is real and \( > 0 \), M. G. de Bruin, E. B. Saff and R. S. Varga have proved in \([1, 5, 7]\) that the result of Theorem 3 is sharp in the sense that each boundary point of \( \{ z = re^{i\theta} \in \mathbb{C} : 0 < r < (1 + \cos \theta)/2, |\theta| < \pi \} \) is an accumulation point of zeros of the normalized Bessel polynomials

\[
Y_{m}(2z/(m + \delta + 1)), \quad m \in \mathbb{N}, \ m + \delta + 1 > 0.
\]

Whether for nonreal \( m + \delta + 1 \), Theorem 3 (and for nonreal \( a_n, b_n \), Corollary 2) still is sharp is an open question. For other new results concerning the location of zeros of polynomials satisfying a three-term recurrence relation see also \([4]\).

**References**


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