ON THE DEGREE OF APPROXIMATION OF A CLASS
OF FUNCTIONS BY MEANS OF FOURIER SERIES

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Abstract. In this paper degree of approximation of Lebesgue integrable functions
by means of Fourier series is examined.

1. Let \( f \) be a periodic function with period \( 2\pi \) and integrable in the sense of
Lebesgue. Let
\[
 f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx).
\]
We write
\[
 \phi(t) = \frac{1}{2} \{ f(x+t) + f(x-t) - 2f(x) \} \quad \text{and} \quad \Phi(t) = \int_0^t |\phi(u)| \, du.
\]
Let
\[
 w_1(\delta) = w_1[f,x](\delta) = \sup_{|h|<\delta} \left\{ \frac{1}{2h} \int_{-h}^{h} |f(x+u) - f(x)| \, du \right\}.
\]
It is clear that for \( f \in C^*[0,2\pi] \), \( w_1(\delta) \leq w[f](\delta) \), where \( w[f](\delta) \) denotes
the modulus of continuity of \( f \).

Let \( \Lambda = (\lambda_{n,k}) \), \( k = 0,1,2,\ldots,n \), be a triangular matrix and let
\[
 \sigma_n = \sum_{k=0}^{n} \lambda_{n,k} s_k,
\]
where \( \{s_k\} \) is a given sequence of numbers. \( \sigma_n \) is called \( n \)th \( \Lambda \)-means of \( \{s_n\} \). If
\( \sigma_n \to s \) as \( n \to \infty \), we say that \( \{s_n\} \) is summable (\( \Lambda \)) to \( s \). We suppose that \( \{\lambda_{n,k}\} \)
is nonnegative with \( \sum_{k=0}^{n} \lambda_{n,k} = 1, n = 0,1,\ldots \). Then a necessary and sufficient
condition for regularity of the \( \Lambda \)-method is \( \lim_{n \to \infty} \lambda_{n,k} = 0 \) for each \( k \).

For
\[
 \lambda_{n,k} = \frac{P_{n-k}}{P_n}, \quad P_n = p_0 + p_1 + \cdots + p_n, p_0 > 0,
\]
the \( \Lambda \)-method reduces to the \((N, p_n)\) method. Similarly for \( \lambda_{n,k} = p_k/P_n \), we get
\((\bar{N}, p_n)\) means.
In what follows we assume that $C$ is a positive constant not necessarily the same at each occurrence.

2. In this paper we establish the following

THEOREM. Suppose for fixed $x$, $w_1(\delta) < \infty$ for $\delta \in (0, \pi]$, and let $\sigma_n(x)$ denote the $n$th $A$-means of the Fourier series of $f(x)$. If $\{\lambda_{n,k}\}$ is nondecreasing with respect to $k$, then

\[
|\sigma_n(x) - f(x)| \leq C \sum_{k=0}^{n} \frac{w_1(\pi/(k+1))}{k+1} \sum_{\nu=0}^{k} \lambda_{n,n-\nu}.
\]

3. PROOF. We have

\[
\sigma_n(x) - f(x) = \frac{2}{\pi} \int_0^{\pi} \phi(t) \sum_{k=0}^{n} \lambda_{n,k} D_k(t) \, dt
\]

\[
= \frac{2}{\pi} \left( \int_0^{\pi/n} + \int_{\pi/n}^{1} \right) = I_1 + I_2, \quad \text{say},
\]

where

\[
D_k(t) = \frac{\sin \left( k + \frac{1}{2} \right) t}{2 \sin \frac{t}{2}}.
\]

Since

\[
\Phi(t) = \int_0^t |\phi(u)| \, du \leq \frac{1}{2} \int_0^t |f(x + u) - f(x)| \, du \leq tw_1(t),
\]

it follows that

\[
|I_1| \leq \frac{2}{\pi} \int_0^{\pi/n+1} \Phi(t) \sum_{k=0}^{n} \lambda_{n,k} \left( k + \frac{1}{2} \right) \, dt
\]

\[
\leq 2w_1 \left( \frac{\pi}{n+1} \right) \leq 2 \sum_{k=0}^{n} \frac{w_1(\pi/(k+1))}{k+1} \sum_{\nu=0}^{k} \lambda_{n,n-\nu}.
\]

Let $\gamma_n$ be a sequence of linear functions on $[k, k+1]$ such that $\gamma_n(k) = \lambda_{n,n-k}$, $k = 0, 1, 2, \ldots, n$, and let $F_n(t) = \int_0^t \gamma_n(u) \, du$, $t \geq 0$. Then

\[
F_n(k) = \sum_{\nu=0}^{k-1} \frac{\gamma_n(v+1) + \gamma_n(v)}{2} = \sum_{\nu=0}^{k-1} \frac{\lambda_{n,n-v-1} + \lambda_{n,n-v}}{2}
\]

\[
\leq \sum_{\nu=0}^{k} \lambda_{n,n-\nu} \leq 2F_n(k).
\]

Using the well-known estimate of McFadden [5],

\[
\left| \sum_{k=a}^{b} \lambda_{n,n-k}e^{i(n-k)\xi} \right| \leq 2(2\pi + 1) F_n \left( \frac{\pi}{t} \right),
\]
where $0 < a \leq b \leq \infty$, $0 < t \leq \pi$ and $n$ is any nonnegative integer, we have

$$|I_2| \leq \frac{2}{\pi} \int_{\pi/n+1}^{\pi} \frac{\phi(t)}{t} \sum_{k=0}^{n} \lambda_{n,k} D_k(t) \, dt \leq C \int_{\pi/n+1}^{\pi} \frac{\phi(t)}{t} F_n^{(\pi)}(\frac{\pi}{t}) \, dt$$

$$= C \left\{ \frac{\Phi(t)}{t} F_n^{(\pi)}(\frac{\pi}{t}) \right\}^{\pi} + \int_{\pi/n+1}^{\pi} \frac{\Phi(t)}{t^2} F_n^{(\pi)}(\frac{\pi}{t}) \, dt$$

$$+ \int_{\pi/n+1}^{\pi} \frac{\Phi(t)}{t} F_n^{(\pi)}(\frac{\pi}{t}) \cdot \frac{\pi}{t^2} \, dt \right\}$$

$$= C \left\{ \frac{\Phi(\pi)}{\pi} F_n(1) - \frac{(n+1)}{\pi} \frac{\Phi(\pi/n+1)}{n+1} F_n(n+1) \right.$$}

$$\left. + \int_{1}^{n+1} \frac{\Phi(\pi/t)}{\pi} F_n(t) \, dt + \int_{1}^{n+1} \frac{\Phi(\pi/t)}{\pi} F_n^{(\pi)}(\frac{\pi}{t}) \, dt \right\}$$

$$\leq C w_1(\pi) \lambda_{n,n} + C \sum_{k=1}^{n} \int_{k}^{k+1} \frac{\pi}{t} w_1(\pi/t) F_n(t) \, dt + C \sum_{k=1}^{n} \int_{k}^{k+1} w_1(\pi/t) \gamma_n(t) \, dt$$

$$= I_{21} + I_{22} + I_{23}, \text{ say.}$$

Obviously,

$$I_{21} \leq C \sum_{k=0}^{n} w_1(\frac{\pi}{k+1}) \lambda_{n,n-k} \leq C \sum_{k=0}^{n} \frac{w_1(\pi/(k+1))}{k+1} \sum_{\nu=0}^{k} \lambda_{n,n-\nu}$$

and

$$I_{22} \leq C \sum_{k=1}^{n} \frac{w_1(\pi/k)}{k} F_n(k+1) \leq C \sum_{k=0}^{n} \frac{w_1(\pi/(k+1))}{k+1} \sum_{\nu=0}^{k} \lambda_{n,n-\nu}$$

$$\leq C \sum_{k=0}^{n} \frac{w_1(\pi/(k+1))}{k+1} \sum_{\nu=0}^{k} \lambda_{n,n-\nu}.$$}

Similarly,

$$I_{23} \leq C \sum_{k=1}^{n} \frac{w_1(\pi/k)}{k} \left( \frac{\gamma_n(k) + \gamma_n(k+1)}{2} \right) \leq C \sum_{k=1}^{n} \frac{w_1(\pi/k)}{k} \lambda_{n,n-k}$$

$$\leq C \sum_{k=0}^{n} \frac{w_1(\pi/(k+1))}{k+1} \lambda_{n,n-k} \leq C \sum_{k=0}^{n} \frac{w_1(\pi/(k+1))}{k+1} \sum_{\nu=0}^{k} \lambda_{n,n-\nu}.$$}

Thus

$$|I_2| \leq C \sum_{k=0}^{n} \frac{w_1(\pi/(k+1))}{k+1} \sum_{\nu=0}^{k} \lambda_{n,n-\nu}.$$}

From (3.1) and (3.2) the proof of our theorem follows.

4. Taking $\lambda_{n,k} = p_{n-k}/P_n$, where $(p_k)$ is a positive nonincreasing sequence, we deduce from (2.1) the following theorem due to Markiewicz [4]; the case $p_n = 1$ is due to Aljančić, Bojanic and Tomić [1].
Corollary 1. If for fixed $x$, $w_1(\delta) < \infty$ for $\delta \in (0, \pi]$, then
\[ |t_n(x) - f(x)| \leq \frac{C}{P_n} \sum_{k=0}^{n} \frac{P_k}{k+1} w_1 \left( \frac{\pi}{k+1} \right), \]
where $t_n(x)$ denotes the $(N, p_n)$ means of the Fourier series of $f(x)$.

This result in weaker form, where $w_1$ is replaced by $w$, is due to Holland, Sahney and Tzimbalario [3]. For related results concerning Cesàro summability see Obrechkoff [6] and Flett [2].

Corollary 2. If $\{\lambda_{n,k}\}$ is a nonincreasing sequence with respect to $k$, then under the hypothesis of the theorem,
\[ |\sigma_n(x) - f(x)| \leq C \sum_{k=0}^{n} \lambda_{n,k} w_1 \left( \frac{\pi}{k+1} \right). \]

Proof. Let $t_n^*(x)$ denote the $(C, 1)$ mean of the Fourier series. Then taking $\lambda_{n,k} = (n+1)^{-1}$ in our Theorem, we have
\[ |t_n^*(x) - f(x)| \leq \frac{C}{n+1} \sum_{k=0}^{n} w_1 \left( \frac{\pi}{k+1} \right). \]

Using a partial summation formula of Abel,
\[ \sigma_n(x) - f(x) = \sum_{\nu=0}^{n} \lambda_{n,\nu} (s_{\nu}(x) - f(x)) \]
\[ = \sum_{\nu=0}^{n-1} \Delta \lambda_{n,\nu} (\nu+1) (t^*_\nu(x) - f(x)) + \lambda_{n,n} (n+1) (t^*_n(x) - f(x)) \]
\[ = \sum_{\nu=0}^{n} (\nu+1) (t^*_\nu(x) - f(x)) \Delta \lambda_{n,\nu}. \]

Since $\Delta \lambda_{n,\nu} \geq 0$ we have, from (4.1),
\[ |\sigma_n(x) - f(x)| \leq C \sum_{\nu=0}^{n} \Delta \lambda_{n,\nu} \sum_{k=0}^{\nu} w_1 \left( \frac{\pi}{k+1} \right) \]
\[ = C \sum_{k=0}^{n} w_1 \left( \frac{\pi}{k+1} \right) \sum_{\nu=k}^{n} \Delta \lambda_{n,\nu} = C \sum_{k=0}^{n} \lambda_{n,k} w_1 \left( \frac{\pi}{k+1} \right). \]

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References