SOME INTEGRAL FORMULAS FOR HYPERSURFACES 
AND A GENERALIZATION OF 
THE HILBERT-LIEBMAN THEOREM

LI AN-MIN

Abstract. R. C. Reilly calculated the variations of functions of the mean curvatures 
for hypersurfaces in Euclidean space. In the present paper, using Reilly's formulas, 
we derive some general integral formulas for hypersurfaces, which generalize 
the well-known Minkowski formulas, and then apply those formulas to obtain some 
characterizations of the hypersphere.

Let \( M \) be a closed hypersurface in \( E^{n+1} \) and \( H_\gamma \) the \( r \)th mean curvature of \( M \), i.e.

\[
H_\gamma = \sum \lambda_1 \lambda_2 \cdots \lambda_\gamma,
\]

where \( \lambda_1, \ldots, \lambda_n \) are the principal curvatures of \( M \). R. C. Reilly proved the following 
formula for the variation \([1]\):

\[
\frac{d}{dt} \int f(H_1, \ldots, H_n) \, dV \bigg|_{t=0} = \int_M \left\{ \frac{\partial f}{\partial H_\gamma} \right( H_1, \ldots, H_n ) \right. \left. + \sum_{\gamma=1}^{n} \left( H_\gamma H_1 - (\gamma + 1) H_{\gamma+1} \right) D_\gamma f(H_1, \ldots, H_n) \right. \\
+ \sum_{\gamma=1}^{n} \left( D_\gamma f(H_1, \ldots, H_n) \right)_{ij} T_{\gamma}^{ij} \right\} \, dV,
\]

where \( f(H_1, \ldots, H_n) \) is any smooth function, \( D_\gamma f(H_1, \ldots, H_n) = \partial f(H_1, \ldots, H_n) / \partial H_\gamma \), 
\( T_{\gamma}^{ij} \) is the Newton tensor, \( y \) is the normal component of the variational vector, and 
\( H_{n+1} \equiv 0 \).

Let \( X \) be the position vector and consider the deformation

\[
M_t: X_t = X + tX = (1 + t)X.
\]

Now we want to compute \( (d/dt) \int_M f(H_1(t), \ldots, H_n(t)) \, dV(t) \bigg|_{t=0} \) directly. Choose a 
local frame field \( e_1, \ldots, e_n, e_{n+1} \) such that \( e_{n+1} \) is the unit normal vector to \( M \) at \( X \). 
From (3) we get

\[
dX_t = (1 + t) \, dX,
\]
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so \( C_{n+1} \) is also the unit normal vector to \( M_t \). Let

\[
dX = \omega_i e_i, \quad dX_i = \omega_i(t) e_i, \quad 1 \leq i \leq n;
\]

we get

\[
(4) \quad \omega_i(t) = (1 + t) \omega_i,
\]

\[
(5) \quad \omega_{n+1,i}(t) = \omega_{n+1,i},
\]

so

\[
(6) \quad dV(t) = (1 + t)^n dV.
\]

For any real parameter \( \lambda \) we have

\[
(7) \quad (\omega_1(t) + \lambda \omega_{n+1,1}(t)) \wedge (\omega_2(t) + \lambda \omega_{n+1,2}(t)) \wedge \cdots \wedge (\omega_n(t) + \lambda \omega_{n+1,n}(t))
\]

\[
= \sum_{\gamma=0}^{n} H_{\gamma}(t) \lambda^\gamma \omega_1(t) \wedge \cdots \wedge \omega_n(t) = \sum_{\gamma=0}^{n} H_{\gamma}(t) (1 + t)^\gamma \omega_1 \wedge \cdots \wedge \omega_n.
\]

On the other hand we have

\[
(8) \quad (\omega_1(t) + \lambda \omega_{n+1,1}(t)) \wedge \cdots \wedge (\omega_n(t) + \lambda \omega_{n+1,n}(t))
\]

\[
= ((1 + t) \omega_1 + \lambda \omega_{n+1,1}) \wedge \cdots \wedge ((1 + t) \omega_n + \lambda \omega_{n+1,n})
\]

\[
= (1 + t)^n \sum_{\gamma=0}^{n} H_{\gamma} \left( \frac{\lambda}{1 + t} \right)^\gamma \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n.
\]

From (7) and (8) we have

\[
(9) \quad H_{\gamma}(t) = H_{\gamma}/(1 + t)^\gamma.
\]

Due to (6) and (9) we get

\[
(10) \quad \frac{d}{dt} \int_M f \left( H_1(t), \ldots, H_\gamma(t) \right) dV(t) \bigg|_{t=0} = \int_M \left( n f - \sum_{\gamma=1}^{n} \gamma D_\gamma f \cdot H_\gamma \right) dV.
\]

Comparing it with (2) we get

**Theorem 1.**

\[
(11) \quad \int_M \left( n f - \sum_{\gamma=1}^{n} \gamma D_\gamma f H_\gamma \right) dV
\]

\[
= \int_M \left\{ -H_1 f + \sum_{\gamma=1}^{n} (H_\gamma H_1 - (\gamma + 1)H_{\gamma+1}) D_\gamma f + \sum_{\gamma=1}^{n} \left( D_\gamma f \right)_{,ij} T_{ij}^{\gamma+1} \right\} dV,
\]

where \( P \) is the support function of \( M \).

Setting \( f = H_\gamma \), (11) gives the well-known Minkowski formulas. Using Theorem 1 we can prove

**Theorem 2.** A closed strictly convex hypersurface in \( E^{n+1} \) with \( H_\gamma = \text{const} \) is a hypersphere.
Proof. Let \( f = H^{n/\gamma} \). (11) gives

\[
0 = \int_M \frac{H^{(n-\gamma)/\gamma}_y}{\gamma} ((n-\gamma)H_y H_{y} - n(\gamma + 1)H_{y+1}) P \, dV 
+ \int_M \frac{n}{\gamma} \left( H^{(n-\gamma)/\gamma}_y \right)_{\gamma} T_{\gamma-1}^{ij} P \, dV.
\]

Due to \( H_y = \text{const} \) we have

\[
\int_M \frac{H^{(n-\gamma)/\gamma}_y}{\gamma} ((n-\gamma)H_y H_{y} - n(\gamma + 1)H_{y+1}) P \, dV = 0.
\]

Choose the origin in \( E^{n+1} \) such that \( P > 0 \); since \( M \) is a strictly convex hypersurface we have

\[
H_y > 0
\]

and [2]

\[
(n-\gamma)H_y H_{y} - n(\gamma + 1)H_{y+1} > 0.
\]

The equality sign holds here if and only if \( \lambda_1 = \lambda_2 = \cdots = \lambda_n \). Hence \( M \) must be a hypersphere.

This theorem is a generalization of the Hilbert-Liebmann theorem.

Furthermore we get the following result from (12).

**Theorem 3.** Let \( M \) be a strictly convex closed hypersurface in \( E^{n+1} \) and the origin an interior point of \( M \). Then we have

\[
\int_M \left( H^{(n-\gamma)/\gamma}_y \right)_{\gamma} T_{\gamma-1}^{ij} P \, dV \leq 0.
\]

The equality sign holds if and only if \( M \) is a hypersphere.

**Theorem 4.** A closed strictly convex hypersurface in \( E^{n+1} \) with \( H_y/H_{\mu} = \text{const} \) (for fixed \( \gamma, \mu \), \( 1 \leq \mu < \gamma < n \), is a hypersphere.

**Proof.** In the integral formula (11) we put

\[
f = \frac{H^{(n-\gamma)/\gamma}_y}{H^{(n-\gamma)/\gamma}_\mu} / H^{(n-\gamma)/\gamma}_\mu / H^{(n-\gamma)/\gamma} / (\gamma - \mu).
\]

Then we get

\[
0 = \int_M \frac{1}{\gamma - \mu} H^{(n-\gamma)/\gamma}_y (n-\gamma)(1 + \mu)H_y H_{y+1}
- (n - \mu)(1 + \gamma)H_y H_{y+1}) P \, dV
+ \int_M \left[ \frac{n - \mu}{\gamma - \mu} \left( \frac{H_y}{H_\mu} \right)_{\gamma} T_{\gamma-1}^{ij} - \frac{n - \gamma}{\gamma - \mu} \left( \frac{H_y}{H_\mu} \right)_{\gamma} T_{\gamma-1}^{ij} \right] P \, dV.
\]
Since $H_\gamma / H_\mu = \text{const}$ we have
\begin{equation}
\frac{1}{\gamma - \mu} \int_M \frac{H_\gamma^{(n-\gamma)/(\gamma-\mu)}}{H_\mu^{(n-\mu)/(\gamma-\mu)}} \left((n - \gamma)(1 + \mu)H_\gamma H_{\mu+1} - (n - \mu)(1 + \gamma)H_\mu H_{\gamma+1}\right) P dV = 0.
\end{equation}

Choose the origin in $E^{n+1}$ such that $P > 0$. For a strictly convex hypersurface the following inequality is valid [2]:
\begin{equation}
\frac{H_{n+1}/\left(\binom{n}{1}\right)}{H_\gamma/\left(\binom{n}{\gamma}\right)} \leq \frac{H_\gamma/\left(\binom{n}{\gamma}\right)}{H_{\gamma-1}/\left(\binom{n}{\gamma-1}\right)} \leq \cdots \leq \frac{H_{\mu+1}/\left(\binom{n}{\mu+1}\right)}{H_\mu/\left(\binom{n}{\mu}\right)},
\end{equation}
so
\begin{equation}
(n - \gamma)(1 + \mu)H_\gamma H_{\mu+1} - (n - \mu)(1 + \gamma)H_\mu H_{\gamma+1} \geq 0.
\end{equation}
The equality sign holds if and only if $\lambda_1 = \lambda_2 = \cdots = \lambda_n$. Hence $M$ must be a hypersphere.

REFERENCES

Department of Mathematics, Szechuan University, Chengtu, China