AN INJECTIVE METRIZATION FOR COLLAPSIBLE POLYHEDRA

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Abstract. In this paper we prove that any finite collapsible polyhedron is injectively metrizable.

A metric space $Y$ is injective if every mapping which increases no distance from a subspace of any metric space $X$ to $Y$ can be extended, increasing no distance, over $X$. Isbell [2] proved that every 2-dimensional collapsible polyhedron admits injective metrics. In this paper we generalize the result to any finite collapsible polyhedron, which answers a part of the problem put forward by Isbell [2, 3].

Let $S$ be a simplicial complex. According to [4], $S$ is called collapsible if there is a sequence of increasing subcomplexes $S_0, S_1, \ldots, S_n$ such that $S_0$ is a point, $S = S_n$ and $S_{i+1} = S_i \cup \{ \Delta_i, \tau \}$, where $\Delta_i$ is an $r_i$-dimensional simplex with an $(r_i - 1)$-dimensional face $\tau$ such that $S_i \cap \{ \Delta_i, \tau \} = \emptyset$, $i = 0, 1, \ldots, n - 1$. The polyhedron $\partial S$ of a (collapsible) complex $S$ is called a (collapsible) polyhedron.

Let $K$ be a cubical complex. $I = [0, 1]$, $I^{n+1} = I^n \times I$. Metrize $K$ as follows: assume that each $k$-cube of $K$ is a copy of $I^k$; define the distance between two points $x, y \in |K|$ so that if $x$ and $y$ are in a common cell, for example, in $|I^k|$, then the distance

$$d(x, y) = \max_i |x_i - y_i|,$$

where $x = (x_1, \ldots, x_k), y = (y_1, \ldots, y_k) \in |I^k|$; otherwise the distance is the length of the shortest path joining them. Obviously, $K$ then is a convex metric space.

Definition 1. Let $K$ be a cubical complex. $Y$ a connected subset of $|K|$. $Y$ is called a generalized cuboid of $K$ if for any cell of $K$, for example, $I^k$, either the intersection $Y \cap |I^k| = \emptyset$, or there are $s_i, t_i$, $0 \leq s_i \leq t_i \leq 1$, $i = 1, \ldots, k$, such that $Y \cap |I^k| = \{(y_1, \ldots, y_k) \in I^k | s_i \leq y_i \leq t_i, i = 1, \ldots, k\}$.

For convenience, write GC for generalized cuboid.

Definition 2. Let $K$ be a cubical complex. $K$ is called collapsible if there are a sequence of subcomplexes $K_0, K_1, \ldots, K_n$ of $K$, and nonempty subcomplexes $L_i$ of $K_i$, $i = 0, 1, \ldots, n$, such that $K_0$ is one point, $K = K_n$, and $K_{i+1} = K_i \cup L_i \times I$, where

$$L_i \times I = \{ c \times \{0\}, c \times I, c \times \{1\} | c \in L_i \}, \quad I = [0, 1],$$

$i = 0, 1, \ldots, n - 1$. Such $K$ is called regular if each $|L_i|$ is a GC of $K_i$. 

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Remark 1. Here for every \( c \in L_i \), we always identify \( c \) and \( c \times \{0\} \). In particular \( I^n = I^n \times \{0\} \subset I^{n+1} \).

Remark 2. Isbell \([2]\) gave a different, rather special definition for collapsible cubical complexes in the 2-dimensional case.

Lemma 1. Let \( S \) be a collapsible simplicial complex. Then \( S \) can be subdivided to a regular collapsible cubical complex \( K \) such that the polyhedron of any subcomplex of \( S \) is exactly the polyhedron of the corresponding subcomplex of \( K \).

Proof. Let the subcomplexes of \( S \), \( S_0, S_1, \ldots, S_n \), and the simplex \( \Delta_i, \tau_i \) be as above. Set \( K_0 = S_0 \). Suppose Lemma 1 is true for \( n = i \); we want to show it is true for \( S_{i+1} = S = S' \cup \{ \Delta_i, \tau_i \} \).

Write \( \partial \Delta_i = \Delta_i - \text{Int} \Delta_i \), by the hypothesis of induction, the polyhedron \( |S_i| \) is subdivided to a regular collapsible cubical complex \( M \), and \( \partial \Delta_i - \text{Int} \tau_i \) is a polyhedron of some subcomplex \( L \) of \( M \). Obviously, there exists a homeomorphism \( f \) of \( (\partial \Delta_i - \text{Int} \tau_i) \times I \) onto \( \Delta_i \) such that \( f(x,0) = x \) for every \( x \in \partial \Delta_i - \text{Int} \tau_i \). By \( f \), one can obtain the cubical subdivision \( M' = M \cup L \times I \) of \( |S'| \).

Consider an arrangement \( c_1, c_2, \ldots, c_m \) of all cubes contained in \( L \) so that \( \dim c_i \leq \dim c_{i+1} \), for \( 1 \leq i \leq m \). Set

\[
M_\alpha = M \cup \{ c_j \times I, c_j \times \{1\} \mid j = 1, 2, \ldots, \alpha \}, \quad \alpha = 0, 1, \ldots, m;
\]

then \( M = M_0 \subset M_1 \subset \cdots \subset M_{m-1} \subset M_m = M \cup L \times I \). Let \( Q_\alpha = c_\alpha \times \{0\} \cup \partial c_\alpha \times I \). It is easy to construct a homeomorphism \( f_\alpha \) of \( |Q_\alpha \times I| \) onto \( |c_\alpha \times I| \), \( \alpha = 1, 2, \ldots, m \), such that

\[
f_\alpha(x, t, 0) = \begin{cases} (x, 0) & \text{if } x \in c_\alpha, t = 0, \\ (x, t) & \text{if } x \in \partial c_\alpha, t \in I. \end{cases}
\]

Let \( P_0 = M_0, P_\alpha = P_{\alpha-1} \cup Q_\alpha \times I, \alpha = 1, \ldots, m \). By construction each \( P_\alpha \) is a cubical subdivision of \( M_\alpha \), and \( |Q_\alpha| \) is clearly a GC of \( P_{\alpha-1} \). So \( M \subset P_1 \subset P_2 \subset \cdots \subset P_m \) is a subsequence of regular cubical complexes. Since \( |P_m| \approx |M_m| \approx |S_{i+1}| \), \( P_m \) is as desired. \( \square \)

To study injective metrization we give some properties of GC.

Lemma 2. Suppose \( L \) is a subcomplex of a cubical complex \( K \), \( |L| \) is GC in \( K \), and projection

\[
P : |K| \cup |L \times I| \to |K|
\]

is given by \( p(x) = x \) for \( x \in |K| \) and \( p(y, t) = y \) for \( (y, t) \in |L| \times I \). Let \( K' = K \cup L \times I \). If \( X \) is a GC of \( K' \), then

(i) \( p(X) \) is a GC of \( K \);

(ii) if \( p(X) \cap |L| = \emptyset, \ X = p(X) \);

(iii) if \( p(X) \cap |L| \neq \emptyset \) and \( X \cap |K| = \emptyset \), then there are \( s_0, t_0 \in I, s_0 < t_0 \), such that \( X = p(X) \times [s_0, t_0] \);

(iv) if \( p(X) \cap |L| \neq \emptyset \) and \( X \cap |K| \neq \emptyset \), i.e. \( X \cap |L| \neq \emptyset \), then there is \( t_0 \in I \) such that \( X = (X \cap |K|) \cup ((p(X) \cap |L|) \times [0, t_0]) \).
Proof. (i) If $|K| \cap X \neq \emptyset$, it is easy to see that $p(X) = |K| \cap X$ is a GC of $K$. If $|K| \cap X = \emptyset$, $p(X) \subset |L|$ and hence $p(X)$ is a GC of $L$. Since $|L|$ is a GC of $K$, so is $p(X)$.

(ii) If $p(X) \cap |L| = \emptyset$, $X \subset |K|$ and hence $X = p(X)$.

(iii) and (iv) follow easily from $X = (X \cap |K|) \cup (X \cap |L \times I|)$ and the following

Claim. If $p(X) \cap |L| = \emptyset$, then there are $s_0, t_0 \in I$ such that

$$X \cap (L \times I) = (p(X) \cap |L|) \times [s_0, t_0].$$

It suffices to show that if $(x, s)$ and $(y, t)$ in $|L| \times I$ are points in $X$, then $(y, s)$ is also a point in $X$. In fact, take a broken line in $X$

$$[(x_0, s_0), (x_1, s_1), \ldots, (x_n, s_n)]$$

such that $(x, s) = (x_0, s_0)$, $(y, t) = (x_n, s_n)$, and $[(x_{i-1}, s_{i-1}), (x_i, s_i)]$ belong to a common cube, $i = 1, \ldots, n$. It successively follows from $(x_0, s_0) \in X$ that $(x, s), (x_s, s), \ldots, (x_n, s) = (y, s)$ are in $X$. \hfill \square

For $r > 0$, nonempty subsets $Y$ of $|K|$ and $X$ of $|K'|$, write

$$B(Y, r) = \{y \in |K| \mid d(y, Y) \leq r\},$$

$$B'(X, r) = \{x \in |K'| \mid d(x, X) \leq r\}.$$ 

Lemma 3. Let $K$ and $L$ be as in Lemma 2. Suppose that for every GC of $K$, $Y$, and $s > 0$, $B(Y, s)$ is a GC of $K$. Then for every GC of $K' = K \cup (L \times I)$, $X$, and $r > 0$, $B'(X, r)$ is a GC of $K'$.

Proof. Let $X$ be a GC of $K'$. The proof conveniently splits into two cases:

Case 1. $X \cap |L| \neq \emptyset$. By (iv) of Lemma 2, there is $t_0 \in I$ such that

$$X = (X \cap |K|) \cup ((p(X) \cap |L|) \times [0, t_0]).$$

Let $B_1 = B'(X \cap |K|, r)$, $B_2 = B'((p(X) \cap |L|) \times [0, t_0], r)$, it is easy to see

$$B'(X, r) = B_1 \cup B_2$$

$$= (B_1 \cap |K|) \cup (B_1 \cap |L \times I|) \cup (B_2 \cap |K|) \cup (B_2 \cap |L \times I|).$$

It is obvious that

$$B_1 \cap |L \times I| \subset B_2 \cap |L \times I|,$$

$$B_1 \cap |K| = B(X \cap |K|, r),$$

and

$$B_2 \cap |L \times I| = (B(p(X), r) \cap |L|) \times [0, t_1],$$

where $t_1 = \min\{t_0 + r, 1\}$. Then $B'(X, r) = B(X \cap |K|, r) \cup ((B(p(X), r) \cap |L|) \times [0, t_1])$. Since $X \cap |K|$, $p(X)$ and $|L|$ are GC of $K$, by the hypothesis, $B(X \cap |K|, r)$ and $B(p(X), r) \cap |L|$ are GC of $K$. So $B'(X, r)$ is a GC of $K$. The proof of Case 1 is complete.

Case 2. $X \cap |L| = \emptyset$. Let $r_0 = d(X, |L|)$. One has

$$B'(X, r) = B'(B'(X, r_0), r - r_0)$$

whenever $r_0 \leq r$. 

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Case 2(a). \( X \cap |K| = \emptyset \). By (iii) of Lemma 2, there are \( s_0, t_0 \in I \) such that \( X = p(X) \times [s_0, t_0] \). If \( r_0 \leq r \), let \( t_2 = \min\{1, t_0 + r_0\} \). Since \( B(p(X), r_0) \) is a GC of \( K \), \( B'(X, r_0) = (B(p(X), r_0) \cap |L|) \times [0, t_2] \) is a GC of \( K \). Now \( B'(X, r_0) \cap |L| \neq \emptyset \), by Case 1, \( B'(X, r) \) is a GC of \( K' \). If \( r_0 > r \), similarly,

\[
B'(X, r) = (B(p(X), r) \cap |L|) \times [s_0 - r, t_1]
\]

is a GC of \( K' \).

Case 2(b). \( X \cap |K| \neq \emptyset \), then \( X \subseteq |K| \). If \( r_0 \leq r \), \( B'(X, r_0) = B(X, r_0) \) has nonempty intersection with \( |L| \). By Case 1, \( B'(X, r) \) is a GC of \( K' \). If \( r_0 > r \), \( B'(X, r) = B(X, r) \) is a GC of \( K' \). \( \square \)

Let \( K \) be a cubical complex. \( K \) is said to have property (P) if any collection of GC of \( K \), \( \{X_{\alpha} | \alpha \in A\} \), such that every couple of its members intersect, has a common point.

**Lemma 4.** Let \( K \) and \( L \) be as in Lemma 2. If \( K \) has the property (P), then \( K' = K \cup L \times I \) also has the property (P).

**Proof.** Let \( \{X_{\alpha} | \alpha \in A\} \) be a collection of GC of \( K' \) such that for each \( \alpha \) and \( \beta \) in \( A \), \( X_{\alpha} \cap X_{\beta} \neq \emptyset \). Then \( \{p(X_{\alpha})\} \) pairwise intersect in \( |K| \), and hence \( \bigcap_{\alpha} p(X_{\alpha}) \neq \emptyset \). We want to show \( \bigcap_{\alpha \in A} X_{\alpha} \neq \emptyset \).

If \( X_{\alpha} \cap |K| \neq \emptyset \) for each \( \alpha \in A \), then

\[
\left( \bigcap_{\alpha} X_{\alpha} \right) \cap |K| = \left( \bigcap_{\alpha} \left( X_{\alpha} \cap |K| \right) \right) = \bigcap_{\alpha} p(X_{\alpha}) \neq \emptyset .
\]

Hence \( \bigcap_{\alpha} X_{\alpha} \neq \emptyset \).

If \( X_{\alpha_0} \cap |K| \neq \emptyset \) for some \( \alpha_0 \in A \), then \( X_{\alpha_0} \subseteq |L| \times I \), and \( p(X_{\alpha}) \cap |L| \neq \emptyset \) for each \( \alpha \in A \). By (iii) and (iv) of Lemma 3, for each \( \alpha \in A \), there are \( s_\alpha, t_\alpha \) such that \( 0 \leq s_\alpha \leq t_\alpha \leq 1 \) and

\[
X_{\alpha} = (X_{\alpha} \cap |K|) \cup \left( \left( p(X_{\alpha}) \cap |L| \right) \times [s_\alpha, t_\alpha] \right).
\]

Set \( s = \sup\{s_\alpha | \alpha \in A\} \), \( t = \inf\{t_\alpha | \alpha \in A\} \). One has \( s \leq t \). In fact, if not, there are \( \alpha_1, \alpha_2 \in A \) such that \( s_{\alpha_1} > t_{\alpha_2} \geq 0 \). Then \( X_{\alpha_1} \cap |K| = \emptyset \). Obviously \( X_{\alpha_1} = (p(X_{\alpha_1}) \cap |L|) \times [s_{\alpha_1}, t_{\alpha_1}] \) does not intersect with \( X_{\alpha_2} = (X_{\alpha_2} \cap |K|) \cup \left( \left( p(X_{\alpha_2}) \cap |L| \right) \times [s_{\alpha_2}, t_{\alpha_2}] \right) \). Contradiction. Then \( \bigcap_{\alpha} X_{\alpha} = \bigcap_{\alpha} p(X_{\alpha}) \times [s, t] \neq \emptyset \). \( \square \)

We have to use an important property of injective metric spaces. That is

**Lemma 5.** Let \( X \) be a metric space. Then \( X \) is injective if and only if \( X \) is convex and any collection of solid spheres in pairwise intersection in \( X \) has a common point.

For proof of Lemma 5 see [1].

Now we can obtain our main conclusion.

**Theorem.** Let \( S \) be a finite collapsible simplicial complex. Then there is a distance function in \( S \) such that \( S \) becomes an injective metric space.

**Proof.** By Lemma 1, \( S \) can be subdivided to a regular collapsible cubical complex \( K \) with its natural metric. Let the sequence of subcomplexes of \( K \),

\[
K_0 \subset K_1 \subset \cdots \subset K_n = K,
\]

and \( L_i \subset K_i, \ i = 0, 1, \ldots, n, \) be as in Definition 2. Because \( |K| \) is convex, using Lemma 5, we need only show that every solid sphere in \( |K|, B(x, r) = \{ y \in |K| \mid d(x, y) \leq r \} \) is a GC of \( K \), and that \( K \) has the property (P).

The proof will be by induction on \( n \). If \( n = 0, K_0 = \) one point, it holds obviously. Suppose it holds for \( n = j \geq 0 \). Then the correctness for \( K_{j+1} = K = K' \cup L_j \times I \) easily follows from Lemma 3 and Lemma 4. \( \square \)

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**BIBLIOGRAPHY**


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