CRITICAL POINTS OF ONE PARAMETER FAMILIES
OF MAPS OF THE INTERVAL

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Abstract. It is shown that some of the periodic phenomena which is well known to
occur for the critical point of the quadratic family \( f_a(x) = ax(1 - x) \) (and other \( C^1 \)
families with a single critical point) occurs for each critical point in \( C^1 \) families with
an arbitrary (possibly infinite) number of critical points. Also, some of the same
behavior occurs in families of maps (which are not necessarily differentiable) where
a critical point has derivative zero on either the left or the right side. A stronger
condition is obtained when the derivative on the right is zero.

This paper is concerned with the periodic behavior of critical points of one
parameter families of maps, \( f_s \), of an interval to itself. We let \( (Df_s)^+(x) \) (resp.
\( (Df_s)^-(x) \)) denote the derivative of \( f_s \) at \( x \) on the right (resp. left) and \( (f_s)'(x) \)
denote the derivative of \( f_s \) at \( x \). We let \( C^1([0,1],[0,1]) \) denote the space of continuously
differentiable maps from \([0,1]\) to itself with the \( C^1 \) (uniform) topology. We will use
the term period to mean least period \([1]\).

Theorem. Let \( F: [a, b] \times [0,1] \rightarrow [0,1] \) be continuous and let \( f_s(x) = F(s, x) \).
Suppose that for each \( s \in [a, b] \) there is a point \( z_s \) in the open interval \((0,1)\) such that
the map \( s \rightarrow z_s \) is continuous, and \( f_s(z_a) = z_a \) while \( f_s(z_b) = 1 \). Suppose also that for
each \( s \in [a, b] \), \( f_s(0) = f_s(1) = 0 \).

(1) Suppose \( (Df_s)^+(z_s) = 0 \) for each \( s \in [a, b] \), and there is a constant \( \gamma > 0 \) such
that \( (f_s)'(x) \) exists for all \( x \in (z_s, z_s + \gamma) \), and \( (Df_s)^+(x) \) varies continuously with \( s \)
and \( x \) for \( s \in [a, b] \) and \( x \in [z_s, z_s + \gamma) \). Then there is a sequence \( (s_n) \) in \((a, b)\) with
\( s_1 < s_2 < s_3 < s_4 < s_5 < \cdots \) such that for each \( n = 1, 2, 3, \ldots \), \( z_{s_n} \) is a periodic point
of \( f_{s_n} \) of period \( n \).

(2) Suppose \( (Df_s)^-(z_s) = 0 \) for each \( s \in [a, b] \), and there is a constant \( \gamma > 0 \) such
that \( (f_s)'(x) \) exists for all \( x \in (z_s - \gamma, z_s) \), and \( (Df_s)^-(x) \) varies continuously with \( s \)
and \( x \) for \( s \in [a, b] \) and \( x \in (z_s - \gamma, z_s) \). Then there is a positive integer \( N \) and an
element \( s_n \) of \((a, b)\) for every integer \( n \) with \( n = 1 \) or \( n \geq N \) such that \( z_{s_n} \) is a periodic point
of \( f_{s_n} \) of period \( n \), and \( s_1 < s_N < s_{N+1} < s_{N+2} < \cdots \).

(3) Suppose that for each \( s \in [a, b] \), \( f_s \in C^1([0,1],[0,1]) \), and the map \( s \rightarrow f_s \) from
\([a, b]\) to \( C^1([0,1],[0,1]) \) is continuous. Suppose also that \( (f_s)'(z_s) = 0 \) for all \( s \). Then
there is a sequence \( (s_n) \) as in (1) and also a sequence \( (t_n) \) in the interval \([s_2, s_3]\) with
\( s_2 = t_1 < t_2 < t_3 < \cdots \) such that for each \( n = 1, 2, 3, \ldots \), \( z_{t_n} \) is a periodic point of \( f_{t_n} \)
of period \( 2^n \).
In (3) we have a $C^1$-continuous family of maps $f_s$ with a family of critical points $z_s$ (which vary continuously with $s$), and the point $(z_s, f_s(z_s))$ moves from below the diagonal to a height of 1. The Theorem asserts that $z_s$ becomes periodic of period 1, 2, 4, 8, 16, 32, ..., 3, 4, 5, ... (in succession). The standard example, $f_s(x) = sx(1 - x)$ where $0 \leq s \leq 4$, satisfies the hypothesis of (3) with $z_s = \frac{1}{2}$. Of course the conclusion of (3) is well known for this example and for similar examples with a single critical point, and, in fact, much more detailed (and quite beautiful) information is known [1]. In these examples the kneading theory [3] determines the dynamics (to a large extent), and the first time $z_s$ becomes periodic it is fixed (i.e. has period 1), the next time $z_s$ is periodic it has period 2, then period 4, etc. The Theorem given here does not assert this property, and in fact this property does not hold for the more general types of examples to which the Theorem applies. For example, the family

$$f_s(x) = 64(s + 1)^2 x (1 - x) ((x - 1/2)^2 + s/4), \quad 0 \leq s \leq 1,$$

satisfies the hypothesis of (3) with $z_s = \frac{1}{2}$. The Theorem asserts that $z_s$ becomes periodic of period 1, 2, 4, 8, 16, 32, ..., 3, 4, 5, ..., in succession, but $z_s$ may become periodic of other periods before 1, between 1 and 2, etc. In this example, it is easy to verify that before becoming fixed, $z_s$ actually does become periodic of other periods.

Note that for a given family of maps there may be several (or infinitely many) families of critical points $z_s$ satisfying the hypothesis of (3). For example, the family

$$f_s(x) = 16sx(1 - x) ((x - 1/2)^2 + s/4), \quad 0 \leq s \leq 1,$$

satisfies the hypothesis of (3), where we may take $z_s = \frac{1}{2} - \sqrt{(1 - s)/8}$, $z_s = \frac{1}{2}$, or $z_s = \frac{1}{2} + \sqrt{(1 - s)/8}$. Another example is the family $f_s$ defined for $0 \leq s \leq 1$ by

$$f_s(x) = \begin{cases} 
16s(-x^2 + \frac{1}{2}x) & \text{if } 0 \leq x \leq \frac{1}{4}, \\
s & \text{if } \frac{1}{4} \leq x \leq \frac{1}{2}, \\
16s(-x^2 + \frac{3}{2}x - \frac{1}{2}) & \text{if } \frac{1}{2} \leq x \leq 1. 
\end{cases}$$

Here, we may take $z_s$ to be any continuous family of points in the interval $[\frac{1}{2}, \frac{3}{2}]$.

Note that in (3), when $z_s$ is periodic, it is automatically stable since $(f_s)'(z_s) = 0$ [1].

An example of a family satisfying (1) is given by

$$f_s(x) = \begin{cases} 
sx/2 & \text{if } 0 \leq x \leq \frac{1}{2}, \\
sx(1 - x) & \text{if } \frac{1}{2} \leq x \leq 1, 
\end{cases}$$

for $0 \leq s \leq 4$. A similar example is given by

$$f_s(x) = \begin{cases} 
(s/4)(1 - \sqrt{1 - 2x}) & \text{if } 0 \leq x \leq \frac{1}{2}, \\
sx(1 - x) & \text{if } \frac{1}{2} \leq x \leq 1, 
\end{cases}$$

for $0 \leq s \leq 4$. In both examples, $z_s = \frac{1}{2}$.

An example of a family satisfying (2) is given by

$$f_s(x) = \begin{cases} 
sx(1 - x) & \text{if } 0 \leq x \leq \frac{1}{2}, \\
s(1 - x)/2 & \text{if } \frac{1}{2} \leq x \leq 1, 
\end{cases}$$
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for $0 \leq s \leq 4$. In this example one can verify that the critical point $z_s = \frac{1}{2}$ is never a periodic point of $f_s$ of period 2. Thus, the conclusion of (1) does not hold with the hypothesis of (2).

We remark that some examples similar to ones described here were studied numerically in [2].

We now proceed to prove the Theorem.

PROOF of (1). By hypothesis, $f_s(z_a) \leq z_a$ and $f_h(z_b) > z_b$. Hence, by continuity of $F$ and $z_s$, for some $s_1 \in (a, b)$, $f_{s_1}(z_s) = z_s$. We may take $s_1$ to be the maximal element of $(a, b)$ with $f_{s_1}(z_s) = z_s$. Then for all $s > s_1$, $f_s(z_s) > z_s$.

Since $(Df_s)^+(z_s) = 0$ and $(Df_s)^+(x)$ varies continuously with $s$ and $x$ for $x \in [z_s, z_s + \gamma)$, there is an open interval $N_1 \subset (a, b)$ with $s_1 \in N_1$ and $\delta > 0$ with $\delta < \gamma$, such that $| (Df_s)^+(x) | < 1$ for all $s \in N_1$ and $x \in [z_s, z_s + \delta)$. Since $(f_s)'(x)$ exists for $x \in (z_s, z_s + \delta)$, $| (f_s)'(x) | < 1$ for $s \in N_1$ and $x \in (z_s, z_s + \delta)$. It follows from the continuity of $F$ and the fact that $f_{s_1}(z_s) = z_s$, while $f_s(z_s) > z_s$ for all $s > s_1$, that there is an open interval $N_2 \subset N_1$ with $s_1 \in N_2$ such that $f_s(z_s) \in (z_s, z_s + \delta)$ for all $s \in N_2$ with $s > s_1$.

Let $s_\lambda \in N_2$ with $s_\lambda > s_1$. Then $f_{s_\lambda}(z_{s_\lambda}) > z_{s_\lambda}$ and $f_{s_\lambda}(z_{s_\lambda}) < z_{s_\lambda} + \delta$. Since $| f'(x) | < 1$ for all $x \in (z_s, z_s + \gamma)$, it follows from the mean value theorem that $(f_{s_\lambda})^2(z_{s_\lambda}) = f_{s_\lambda}(f_{s_\lambda}(z_{s_\lambda})) > z_{s_\lambda}$. Since $(f_{s_\lambda})^2(z_{s_\lambda}) = 0 < z_{s_\lambda}$, it follows by continuity of $F$ that for some $s_2 \in (s_\lambda, b)$, $(f_{s_2})^2(z_{s_\lambda}) = z_{s_2}$. Since $s_2 > s_1$, $f_{s_2}(z_{s_\lambda}) \neq z_{s_\lambda}$, so $z_{s_\lambda}$ is a periodic point of $f_{s_\lambda}$ of period 2. We may take $s_2$ to be the maximal element of $(s_\lambda, b)$ with $(f_{s_2})^2(z_{s_\lambda}) = z_{s_2}$.

Now $(f_{s_2})^3(z_{s_\lambda}) = f_{s_2}(z_{s_\lambda}) > z_{s_2}$, and the conclusion of (1) follows by repeating the argument of the preceding paragraph inductively.

PROOF of (2). As in the proof of (1) for some $s_1 \in (a, b)$ (which is chosen to be maximal), $f_{s_1}(z_s) = z_s$. Then $f_s(z_s) > z_s$ for all $s \in (s_1, b)$.

If for some $s_\lambda \in (s_1, b)$, $(f_{s_\lambda})^2(z_{s_\lambda}) = z_{s_\lambda}$, then the proof of (1) applies and the conclusion follows. This is true because the hypotheses concerning differentiability were only used in the proof of (1) to produce such a parameter $s_\lambda \in (s_1, b)$. Hence, we may assume that $(f_{s_\lambda})^2(z_{s_\lambda}) < z_{s_\lambda}$ for all $s \in (s_1, b)$.

Since $(Df_{s_\lambda})^-(z_{s_\lambda}) = 0$ and $(Df_{s_\lambda})^-(x)$ varies continuously with $s$ and $x$ for $x \in (z_s - \gamma, z_s)$, there is an open interval $N_1 \subset (a, b)$ with $s_1 \in N_1$ and $\delta > 0$ with $\delta < \gamma$ such that if $s \in N_1$ and $x \in (z_s - \delta, z_s)$, then $| (Df_{s_\lambda})^-(x) | < 1$. Since $(f_{s_\lambda})'(x)$ exists for all $x \in (z_s - \delta, z_s)$, $| (f_{s_\lambda})'(x) | < 1$ for $s \in N_1$ and $x \in (z_s - \delta, z_s)$. It follows from the continuity of $F$ and the fact that $(f_{s_\lambda})^2(z_{s_\lambda}) = z_{s_2}$, while $(f_{s_\lambda})^2(z_{s_\lambda}) < z_{s_2}$ for all $s > s_1$, that there is an open interval $N_2 \subset N_1$ with $s_1 \in N_2$ such that if $s \in N_2$ with $s > s_1$, then $(f_{s_\lambda})^2(z_{s_\lambda}) \in (z_s - \delta, z_s)$.

Let $s_\lambda \in N_2$ with $s_\lambda > s_1$. Then $| (f_{s_\lambda})'(x) | < 1$ for all $x \in ((f_{s_\lambda})^2(z_{s_\lambda}), z_{s_\lambda})$. Since $f_{s_\lambda}(z_{s_\lambda}) > z_{s_\lambda}$, it follows from the mean value theorem that

\[
(f_{s_\lambda})^3(z_{s_\lambda}) = f_{s_\lambda}((f_{s_\lambda})^2(z_{s_\lambda})) > (f_{s_\lambda})^2(z_{s_\lambda}).
\]
We claim that for some integer \( n > 2 \), \((f^S_x)^n(z^S_x) \geq z^S_x\). To prove this, suppose that for all \( n > 2 \), \((f^S_x)^n(z^S_x) < z^S_x\). It follows from the mean value theorem that

\[
(f^S_x)^2(z^S_x) < (f^S_x)^3(z^S_x) < (f^S_x)^4(z^S_x) < \cdots < z^S_x.
\]

Let \( y^S_x \) be the limit of the increasing sequence \((f^S_x)^n(z^S_x) (n = 2, 3, 4, \ldots)\). Then \( f^S_x(y^S_x) = y^S_x \) and \( f^S_x(z^S_x) > z^S_x \) contradicts, by the mean value theorem, the fact that \(|(f^S_x)(x)| < 1\) for all \( x \in (z^S_x - \delta, z^S_x) \). This establishes our claim that for some integer \( n > 2 \), \((f^S_x)^n(z^S_x) \geq z^S_x\). Since \((f^S_x)^n(z^S_x) < z^S_x\), for some \( t \in (s^x, b) \), \((f^S_x)^n(z^S_x) = z^S_x\). We let \( N \) denote the period of \( z^S_t \) (under \( f^S_x \)) and set \( s_N = t \). Note that \( N > 1 \) since \( t > s^x \), and \( f^S_x(z^S_x) > z^S_x \) for all \( s > s^x \). We may assume, by choosing \( t \) larger and \( N \) smaller if necessary, that \( t \) is the maximal element of \((s^x, b) \) such that \( z^S_t \) is a periodic point of \( f^S_x \) of period \( k \), where \( 2 < k < N \). Now, repeating the argument at the end of the proof of (1), we obtain \( s_N \) for each integer \( n \geq N \) with \( s_N < s_{N+1} < s_{N-2} < \cdots \) such that for each \( n \geq N \), \( z^S_x \) is a periodic point of \( f^S_{s^N} \) of period \( n \).

**Proof of (3).** Since the hypothesis of (3) is stronger than the hypothesis of (1) we obtain \( s_1 < s_2 < s_3 \cdots \) as in the proof of (1).

Note that by the chain rule, \(((f^S_x)^2)'(z^S_x) = 0\) and, so, there are neighborhoods \( N_1 \) of \( s_1 \) in \((a, b)\) and \( N_2 \) of \( z^S_x \) in \((0, 1)\) such that \(|((f^S_x)^2)'(x)| < 1\) if \( s \in N_1 \) and \( x \in N_2 \). Furthermore, there is a neighborhood \( N_3 \) of \( s_2 \) with \( N_1 \subset N_3 \) such that if \( s \in N_3 \), then \( z^S_x \in N_2 \) and \(((f^S_x)^2)'(z^S_x) < z^S_x \). Let \( s^x \in N_3 \) with \( s^x > s_2 \). Then \(((f^S_x)^2)'(z^S_x) < z^S_x \) (by choice of \( s_2 \)) and \(((f^S_x)^2)'(z^S_x), z^S_x) \subset N_2 \). Thus, \(|((f^S_x)^2)'(x)| < 1\) for all \( x \in [(f^S_x)^2(z^S_x), z^S_x] \). By the mean value theorem (applied to \((f^S_x)^2\)),

\[
(f^S_x)^4(z^S_x) = (f^S_x)^2((f^S_x)^2(z^S_x)) < z^S_x.
\]

Since \((f^S_x)^4(z^S_x) = f^S_x(z^S_x) > z^S_x\), for some \( t_2 \) with \( s_2 < t_2 < s_3 \), \((f^S_x)^4(z^S_x) = z^S_t \). By choice of \( s_2 \), \( z^S_t \) is a periodic point of \( f^S_x \) of period 4. We may take \( t_2 \) to be the largest element of \((s_2, s_3)\) such that \((f^S_x)^4(z^S_x) = z^S_t \). Then \((f^S_x)^4(z^S_x) > z^S_x \) for all \( s \) with \( t_2 < s \leq s_3 \).

Now, for \( t \) slightly larger than \( t_2 \), \((f^S_x)^4(z^S_x) \) is slightly larger than \( z^S_t \). Hence, \((f^S_x)^4(z^S_x) \) is also larger than \( z^S_t \) (the proof uses the mean value theorem, as above, applied to \((f^S_x)^4\)). Since \((f^S_x)^8(z^S_x) = (f^S_x)^2(z^S_x) < z^S_x \) (as \((f^S_x)^2(z^S_x) < z^S_x \) for all \( s > s_2 \)), for some \( t_3 \) with \( t_2 < t_3 < s_3 \), \((f^S_x)^8(z^S_x) = z^S_t \). By choice of \( t_2 \), \( z^S_t \) is a periodic point of \( f^S_x \) of period 8.

The conclusion of (3) follows by repeating the argument of the preceding two paragraphs inductively.

**References**

