MULTIPLIER REPRESENTATIONS OF DISCRETE GROUPS

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Abstract. Let \( \sigma \) be a multiplier on the discrete group \( G \). Extending theorems of Kaniuth and Thoma to the case of multiplier representations, we determine when the left regular \( \sigma \) representation of \( G \) has a type I subrepresentation, and when all the \( \sigma \) representations of \( G \) are type I.

1. Let \( \sigma \) be a normalized multiplier on the discrete group \( G \) (cf. [1, 7, 9]). An element \( x \in G \) is \( \sigma \)-regular if \( \sigma(x, a) = \sigma(a, x) \) for all \( a \) which commute with \( x \). If \( x \) is \( \sigma \)-regular so is every conjugate of \( x \) [7, Lemma 3], and we may speak of the \( \sigma \)-regular conjugacy classes in \( G \). Let \( \Delta_0 = \Delta_0(\sigma) \) be the set of elements lying in finite \( \sigma \)-regular conjugacy classes in \( G \). Let \( \Delta_0 = \Delta_0(\sigma) \) be the set of elements lying in finite \( \sigma \)-regular conjugacy classes, let \( \Delta = \Delta(\sigma) \) be the subgroup generated by \( \Delta_0 \), and let \( \Delta' \) be its commutator subgroup. The left regular \( \sigma \) representation \( \lambda_{\sigma} = \lambda_{G,\sigma} \) acts on \( l^2(G) \) by

\[
\lambda_{\sigma}(x)f(y) = \sigma(x^{-1}, y)f(x^{-1}y), \quad f \in l^2(G), \; x, \; y \in G.
\]

It is a finite representation [7, Theorem 9]. We say that the multiplier \( \sigma \) is type I if all the primary \( \sigma \) representations are type I.

Theorem 1. \( \lambda_{\sigma} \) has a type I subrepresentation if and only if the following three conditions are satisfied:

(a) \( [G : \Delta] < \infty \).
(b) \( |\Delta'| < \infty \).
(c) \( \sigma \) is trivial on some cofinite subgroup.

Theorem 2. The following are equivalent:

(a) \( \lambda_{\sigma} \) is type I.
(b) \( \sigma \) is type I.
(c) \( \sigma \) is trivial on some cofinite abelian subgroup.

For the case of ordinary representations (\( \sigma = 1 \)), Theorem 1 is due to Kaniuth [6] and Theorem 2 to Thoma [15, 16]. Smith [13] gave a global proof of these theorems, and the proof of the necessity of (a) and (b) in Theorem 1 is essentially her proof. Though it is quicker to use Thoma’s theorem in the proof of Theorem 2, we indicate another method which gives an alternate proof of Thoma’s theorem. Duflo [3] has also found a version of Theorem 2. After completing most of this work, I learned that A. K. Holzherr [4] had also found, by somewhat different methods, Theorem 2,
most of Theorem 1 and a number of other related results. In particular, he identified
the projection on the type I part of $\lambda_a$. I am indebted to the referee for pointing out
the correct formulation of Theorem 1.

2. For each subset $S \subset G$ we denote by $Z(S) = Z_G(S)$ its centralizer in $G$ and we
put

$$Z_\sigma(S) = \{ y \in Z(S) \mid \sigma(x, y) = \sigma(y, x), \text{all } x \in S \}. $$

If $S = \{x\}$ we write shortly $Z(x)$ or $Z_\sigma(x)$. $x$ is $\sigma$ regular if and only if
$Z_\sigma(x) = Z(x)$. Note that $x \in Z_\sigma \Leftrightarrow y \in Z_\sigma(x)$ and $Z_\sigma(S) = \bigcap_{x \in S} Z_\sigma(x)$.

**Lemma 1.** (a) For each subset $S \subset G$, $Z_\sigma(S)$ is a subgroup.
(b) If $H$ is a finitely generated subgroup of $\Delta$ then $[G : Z_\sigma(H)] < \infty$.

**Proof.** Osima [11] observed that $y \to \sigma(x, y)\sigma(y, x)^{-1}$, $y \in Z(x)$, is a character of
$Z(x)$. $Z_\sigma(x)$ is the kernel of this character. Thus $Z_\sigma(x)$ and $Z_\sigma(S)$ are subgroups. It
is easy to check that $Z_\sigma(a) \cap Z_\sigma(b) \subset Z(ab)$. If $a \in \Delta_0$, $Z_\sigma(a) = Z(a)$ is cofinite. If
$x = a_1 a_2 \cdots a_n \in \Delta$, then $\bigcap_{1}^n Z_\sigma(a_i) \subset Z_\sigma(x)$. Thus $Z_\sigma(x)$ is cofinite. If $H$
generated by $\{x_1, \ldots, x_m\} \subset \Delta$ and $y \in H$, then $\bigcap Z_\sigma(x_i) \subset Z_\sigma(y)$, and $\bigcap Z_\sigma(x_i) \subset
\bigcap_{y \in H} Z_\sigma(y) = Z_\sigma(H)$ is also cofinite.

3. To prove that (a), (b), (c) in Theorem 1 are necessary, we begin by observing
that if the set $\Delta_0$ is a subgroup of infinite index, then by [7, Theorem 4] $\lambda_a$ is type II.
However, the proof of that theorem is valid also in the case that the subgroup $\Delta$
generated by $\Delta_0$ is of infinite index. Thus in what follows we shall assume $\Delta$
is cofinite. Let $\mathcal{U}(G)$ be the bicommutant of $\Lambda_\sigma(G)$, and if $E$ is a projection in $\mathcal{U}(G)$,
let $\mathcal{U}_E(G)$ be the corresponding reduced algebra.

**Lemma 2 (Smith [13]).** Let $G$ be an FC group, $H$ a subgroup, $E$ a central projection
in $\mathcal{U}(G)$ so that $\mathcal{U}_E(G)$ and $\mathcal{U}_E(H)$ are type $1_n$, for some $n < \infty$. Then $[Z_\sigma(H), G]$ is
finite.

Smith's proof of this lemma [13, Lemma 2] for the case $\sigma = 1$ may be adopted,
after replacing the group algebra by the appropriate twisted group algebra, almost
without change to the present setting.

Observing that $\Delta$ is contained in the subgroup of elements lying in finite
conjugacy classes, we see that the remainder of Smith's proof for the case $\sigma = 1$ [13,
pp. 403–404] that if $\lambda_a$ has a type I summand, then $|\Delta'| < \infty$, is valid for arbitrary
$\sigma$. This proves the necessity of (a) and (b) in Theorem 1.

To show there is a cofinite subgroup of $G$ on which $\sigma$ is trivial, it is enough to show
there is a cofinite subgroup of $\Delta$ on which $\sigma$ is trivial. Thus we may suppose
$G = \Delta$ and $[G, G] = G'$ is finite. Put $H = Z_\sigma(G')$ and $F = G' \cap H$. By Lemma 1, $H$
is cofinite. $F$ is finite and $[H, H] \subset F \subset \text{center}(H)$.

**Lemma 3.** $\sigma|_{H \times H}$ is similar to a multiplier lifted from a multiplier $\sigma'$ on $H/F$.

**Proof.** If $\gamma$ is an irreducible $\sigma$ representation of $H$, then for all $a \in F$ and $x \in H$,

$$\gamma(a)\gamma(x) = \sigma(a, x)\gamma(ax) = \sigma(x, a)\gamma(xa) = \gamma(x)\gamma(a).$$
Hence $y(a)$ is a multiple $\varphi(a)$ of the identity, i.e., $y(a) = \varphi(a) \cdot 1$, where $|\varphi(a)| = 1$. Thus $F \subseteq$ projective kernel of $\gamma$ [2, §1], and $\sigma|_{H \times H}$ is similar to a multiplier lifted from a multiplier $\sigma'$ on $H/F$ [2, Lemma 1.3].

Replacing $\sigma|_{H \times H}$ by that multiplier, we may suppose $\sigma(x, y) = \sigma(\bar{x}, \bar{y})$, all $x, y \in H$, where $\bar{x} = xF$.

Because $\lambda_\sigma$ is finite and has a type I part, it contains a finite-dimensional subrepresentation, all of whose matrix coefficients vanish at $\infty$. The restriction of this representation to $H$ contains an irreducible subrepresentation $\tau$, all of whose matrix coefficients vanish at $\infty$. If $n = \text{dim } \tau$, then $\sigma''(x, y) = \text{det} \tau(x) \text{det} \tau(y) \text{det} \tau(xy)^{-1}$. Thus we may replace $\sigma$ by a similar multiplier and assume $\sigma'' = 1$; that is, we may assume $\sigma$ takes its values in a finite subgroup $C$ of the circle.

Consider now the extension $1 \rightarrow C \rightarrow H'' \rightarrow H \rightarrow 1$ of $C$ by $H$ defined by $\sigma$. Put $\pi^0(t, x) = t\pi(x), x \in H, t \in C$. Then $\pi^0$ is a representation of the discrete group $H''$, all of whose matrix coefficients vanish at $\infty$. By a theorem of Taylor [14, Theorem 1, Corollary 1], there is a finite normal subgroup $K$ of $H''$ such that $H''/K$ is a Moore group (only finite-dimensional unitary representations) and $K \subseteq \ker \pi^0$, i.e., $\pi^0$ is lifted from a representation of $H''/K$. $H''/K$ is, in particular, type I, and by Thoma's theorem [13, 15, 16] there is a cofinite subgroup $M \subset H''$ so that $M/K$ is abelian. Then $P = M/K \cap (H/H \cap K)$ is a cofinite abelian subgroup of $H/H \cap K \subset H''/K$. Since $K \subset \ker \pi^0, K \cap H \subset \ker \pi \subset \text{projective kernel of } \pi$. Thus $\sigma|_{H \times H}$ is similar to a multiplier lifted from a multiplier $\sigma''$ of $H/H \cap K$ [2, Lemma 1.3]. Again replacing $\sigma$ by a similar multiplier, we may think of $\pi$ as a $\sigma''$ representation of $H/H \cap K$. Then $\pi|_P$ is a finite-dimensional $\sigma''$ representation of the abelian group $P$. Hence there is a cofinite subgroup $Q \subset M, H \cap K \subset Q$, so that $\sigma''$ is trivial on $Q/H \cap K$ [2, Lemma 2.1, Theorem 3.1]. Then $\sigma|_{Q \times Q}$ is trivial. We have the inclusion $Q \subset M \subset H \subset G$, and since each subgroup is of finite index in the next, $Q$ is cofinite in $G$.

This completes the proof of the necessity of the conditions in Theorem 1. To prove that they are sufficient we begin with

**Lemma 4.** Let $H$ be a subgroup of finite index in $G$ and $\pi$ a finite type I $\sigma$-representation of $H$. Then $\sigma\text{-Ind}_H^G \pi$ is type I.

(For the definition of $\sigma$-induction see [2 or 9].)

**Proof.** $H$ contains a cofinite normal subgroup $N$ and $\pi|_N$ is a finite type I representation. By the Mackey machine, $\sigma\text{-Ind}_N^G \pi|_N$ is also a finite type I representation, and it is easy to see that $\sigma\text{-Ind}_H^G \pi \subset \sigma\text{-Ind}_N^G \pi|_N$.

Let $Q$ be a cofinite subgroup of $G$ on which $\sigma$ is trivial. Then $\Delta \cap Q' = \Delta_Q$ is cofinite, $|\Delta_Q| < \infty$, and because $\Delta$ is contained in the subgroup of $G$ of elements in finite conjugacy classes, each $x \in \Delta_Q$ lies in a finite conjugacy class. By Kaniuth's theorem $\lambda_{\Delta_Q}$ (the left regular representation of $\Delta_Q$) has a type I part. By Lemma 4, $\lambda_0 = \sigma\text{-Ind}_{\Delta_Q}^G \lambda_\sigma$ has a type I part. This completes the proof of Theorem 1.

This theorem can be presented in slightly different terms. For any group $H$ let $H_{FC}$ be the subgroup of elements lying in a finite conjugacy class. Note that $\Delta \subset G_{FC}$.
**Theorem 1'.** \( \lambda_\sigma \) has a type I part if and only if the following three conditions are satisfied:

(a') \([G : G_{FC}] < \infty \).

(b') \(|G'_{FC}| < \infty \).

(c') \( \sigma \) is trivial on some cofinite subgroup.

In fact, suppose \( \lambda_\sigma \) has type I part. Then \([G : \Delta] < \infty \) and because \( \Delta \subset G_{FC} \), \([G : G_{FC}] < \infty \). Since \( |\Delta| < \infty \) and \([G_{FC} : \Delta] < \infty \), \(|G'_{FC}| < \infty \) [10, Lemma 4.1], and by Theorem 1, (c) is satisfied. Conversely, if \( \sigma \) is trivial on a cofinite subgroup \( Q \), then \( Q_{FC} = Q \cap G_{FC} \) is cofinite in \( Q \), and \( Q'_{FC} \subset G'_{FC} \) is also finite. By Kaniuth's theorem [6, 13], \( \lambda_Q \) has a type I part, and by Lemma 4, \( \lambda_\sigma \) has a type I part.

4. The implication (c) \( \Rightarrow \) (b) of Theorem 2 is a consequence of the Mackey machine [2, 9]. In fact, if \( H \) is a cofinite normal subgroup for which \( \sigma|_{H \times H} \) is type I, e.g., an abelian subgroup on which \( \sigma \) is trivial, it follows from the Mackey machine that \( \sigma \) is type I, and every cofinite subgroup contains a cofinite normal subgroup. (b) \( \Rightarrow \) (a) is clear (see [12] for the case of uncountable groups). Thus what remains to show is (a) \( \Rightarrow \) (c).

**Lemma 5.** If \( \lambda_{\sigma,G} \) is type I so is \( \lambda_{\sigma,H} \) for every subgroup \( H \).

This lemma is, in fact, valid for open subgroups of a locally compact group and is what is proved by Kallman in the proof of Proposition 2.4 of [5].

To complete the proof of Theorem 2 we need only observe that if \( Q \) is a cofinite subgroup of \( G \) on which \( \sigma \) is trivial, then \( \lambda_{\sigma,Q} = \lambda_Q \) is type I by Lemma 5, and by Thoma's theorem [13, 15, 16] \( Q \) contains a cofinite abelian subgroup.

5. It is possible to show (a) \( \Rightarrow \) (c) in Theorem 2 without use of Thoma's theorem and obtain, in particular, another proof of that theorem. Assume \( \lambda_\sigma \) is type I. Then by Theorem 1, \( \Delta \) is cofinite and \( \Delta' \) is finite. (This part of Theorem 1 did not use Thoma's theorem.) It is enough to show \( \Delta \) contains a cofinite abelian subgroup on which \( \sigma \) is trivial. Thus we may suppose that \( G = \Delta \) and \( G' = \Delta' \) is finite. As in \( \S 2 \) we put \( H = Z\sigma(G') \), \( F = H \cap G' \) and assume that \( \sigma(x, y) = \sigma'(\tilde{x}, \tilde{y}) \), \( x, y \in H \), where \( \tilde{x} = xF \). Let \( \rho: H \to H/F \) be the canonical map.

We are assuming that \( \lambda_{\sigma,G} \) is type I. By Lemma 5, \( \lambda_{\sigma,H} \) is type I. \( \lambda_{\sigma',H/F} \circ \rho \) is a subrepresentation of \( \lambda_{\sigma,H} \) (the map \( f \to |F|^{-1} f \circ \rho \) is the intertwining operator). Thus \( \lambda_{\sigma',H/F} \) is also type I. But \( H/F \) is abelian and we know [2, Theorem 3.1, Lemma 3.1] that \( \lambda_{\sigma',H/F} \) is type I if and only if there is a cofinite subgroup \( K \subset H \) so that \( \sigma' |_{K/F, K/F} \) is trivial, or what is the same, \( \sigma |_{K \times K} \) is trivial. Replacing \( \sigma \) by a similar multiplier, we may suppose \( \sigma |_{K \times K} = 1 \). Then \( \lambda_{\sigma,K} = \lambda_K \) is also type I (Lemma 5).

We now have an extension \( 1 \to F \to K \to P \to 1 \), where \( F \) is finite and central in \( K \), \( P \) is abelian, and \( K \) has a type I regular representation. We apply the Mackey machine to describe \( \lambda_K \). We have

\[ \lambda_K = \text{Ind}^K_F \lambda_F = \bigoplus_{\theta \in \hat{F}} \text{Ind}^K_F \theta = \bigoplus_{\theta \in \hat{F}} \pi_\theta, \]

where \( \pi_\theta = \text{Ind}^K_F \theta \). Thus \( \lambda_K \) is type I if and only if each \( \pi_\theta \) is type I. Since \( K \) centralizes \( F \), each \( \theta \in \hat{F} \) may be extended to a projective character \( \theta' \) with Mackey
obstruction $\omega^{-1}$, and $\pi_\theta = \theta' \otimes \rho_\theta''$, where $\rho_\theta$ is the right regular $\omega_\theta$ representation of $P$ (here we think of $\omega_\theta$ as a multiplier on $P$), and $\rho_\theta''$ is its lift to $K$ (see [1]). Thus each $\pi_\theta$ is type I if and only if each $\rho_\theta$ is type I, and this happens if and only if $\omega_\theta$ is trivial on a cofinite subgroup $Q_\theta \subset P$ [2, Theorem 3.1, Lemma 3.1]. Let $C$ be the inverse image in $K$ of $\bigcap_{\theta \in F} Q_\theta$. $C$ is cofinite. We shall show it is abelian. Let $\pi \in \hat{C}$. Since

$$F \text{ is central in } C, \quad \pi(a) = \theta_a(a) \cdot 1, \text{ where } \theta_a \in \hat{F}, \, a \in F.$$  

Then $x \to \theta_a^{-1}(x)\pi(x)$ is constant on the $F$ cosets in $C$ and defines a representation $\pi^\star$ of $C/F \subset P$ with multiplier $\omega_\theta$. Since $\omega_\theta$ is trivial on $C/F$, $\omega_\theta(x, y) = \omega_\theta(y, x)$, all $x, y \in C$ and $\pi^\star(x)\pi^\star(y) = \pi^\star(y)\pi^\star(x)$, all $x, y \in C/F$. Since $\pi^\star$ is also irreducible, $\dim \pi^\star = 1 = \dim \pi$. Thus all irreducible representations of $C$ are 1-dimensional and $C$ is abelian.

REFERENCES

5. R. Kallman, Certain topological groups are type I, II, Adv. in Math. 10 (1973), 221–255.