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NOETHERIAN SUBSETS OF PRIME SPECTRA

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ABSTRACT. If X is a noetherian subspace of Spec R, the set of primes of $R[x]$ lying over X is also noetherian. A simple consequence is the theorem of Ohm and Pendleton that a ring module-finite over a j-noetherian ring is j-noetherian.

Let R be a commutative ring with identity, X an arbitrary subspace of Spec R, and T a commutative R-algebra. Our main result is ultimately a version of the Ritt basis theorem [**4**, p. 14] and [**3**, Theorem 7] as generalized by Kaplansky [**1**, Theorem 7.1]. It was developed as a generalization of Theorem 2.5 of Ohm and Pendleton [**2**] in which they show that Spec T is noetherian if Spec R is noetherian and T is finitely generated as an R-algebra. Our proof is adapted from theirs. We also show that Theorem 3.6 of [**2**] is a simple consequence of our theorem.

Let R be a (commutative) ring, X a subspace of Spec R, and I an ideal of R. The *X-radical* of I is the intersection of the elements of X which contain I. I is *X-finite* if it has the same X-radical as some finitely-generated ideal I_0 contained in I. It is not difficult to check that X is *noetherian* (a.c.c. on open subsets) if and only if every ideal is X-finite (see [**2**, Proposition 2.1]). Note that it is possible for every ideal of a ring to have the same X-radical as some finitely-generated ideal without X being noetherian. See [**2**, Example 2.10].

THEOREM. *Let R be a ring and x an indeterminate. Let X be a noetherian subspace of* Spec R. *Let Y be the set of all prime ideals of* $R[x]$ *whose contractions to R are in X. Then Y is noetherian.*

PROOF. Suppose that Y is not noetherian. Let P be maximal among the ideals of $R[x]$ which are not Y-finite. It is routine to show that P must be prime (see [**2**, Proposition 2.3]). We will show that P must in fact be Y-finite; this contradiction will prove the Theorem.

Write P_0 for $P \cap R$. Since P_0 is X-finite, $P_0R[x]$ is Y-finite. Let T be the ring $R[x]/P_0R[x]$. Let $\bar{P}$ be the image of P in T, and let $\bar{Y}$ be the image of Y in Spec T. It is easy to verify that P is Y-finite if and only if $\bar{P}$ is $\bar{Y}$-finite.

We may assume then that P_0 is the zero ideal so that R is an integral domain. Let K be its quotient field. Then $PK[x]$ is a principal ideal; we may assume that it is generated by a nonzero element f of P. If s is the leading coefficient of f, division

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shows that $PR_s[x]$ is already principal (and generated by f). Now s is not in P, so (s, P) is a Y-finite ideal. We can choose elements $a_1, \ldots, a_n$ in $R[x]$ and $b_1, \ldots, b_n$ in P so that $I_0 = (a_1 s + b_1, \ldots, a_n s + b_n)$ has the same Y-radical as (s, P). Let I be the ideal generated by f and the b_i's. I is contained in P; we claim I has the same Y-radical as P.

Suppose that Q is an element of Y which contains I. If s is in Q then I_0 is contained in Q, so that (s, P) is contained in Q. If s is not in Q then $QR_s[x]$ is a prime ideal of $R_s[x]$ which contains f and therefore contains $PR_s[x]$. In either case Q contains P and so P is Y-finite.

COROLLARY. *Let* $\varepsilon\colon R \to T$ *be a homomorphism of rings which makes* T *a finitely-generated* R*-algebra. Let* X *be a noetherian subspace of* $\operatorname{Spec} R$, *and let* Y *be* $\{Q \in \operatorname{Spec} T \mid \varepsilon^{-1}(Q) \in X\}$. *Then* Y *is noetherian.*

PROOF. By induction; subspaces of noetherian spaces are noetherian.

COROLLARY. *Let* R *and* R' *be rings such that* R' *is a finitely-generated* R*-module. If* R *is j-noetherian then so is* R'.

PROOF. Any maximal ideal of R' contracts to a maximal ideal of R.

REFERENCES

1. I. Kaplansky, *An introduction to differential algebra*, Publ. Inst. Math. Univ. Nancago, Paris, 1957.

2. J. Ohm and R. L. Pendleton, *Rings with noetherian spectrum*, Duke Math. J. **35** (1968), 631–640.

3. W. H. Raudenbush, Jr., *Ideal theory and algebraic differential equations*, Trans. Amer. Math. Soc. **36** (1934), 361–368.

4. J. F. Ritt, *Differential equations from the algebraic viewpoint*, Amer. Math. Soc. Colloq. Publ., vol. 14, Amer. Math. Soc., Providence, R.I., 1932.

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