PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 88, Number 3, July 1983

A GEOMETRIC CHARACTERIZATION OF N^+ DOMAINS

PATRICK AHERN AND WILLIAM COHN

ABSTRACT. A connected open set $\mathcal{O} \subseteq \mathbf{C}$ is called an N^+ domain if every holomorphic function defined in the unit disc and taking values in $\mathcal{O}$ is necessarily in the Smirnov class N^+. We show that $\mathcal{O}$ is an N^+ domain if and only if ∞ is a regular point for the solution of the Dirichlet problem for $\mathcal{O}$. We get a similar characterization when N^+ is replaced by the class of outer functions.

Let X be a set of functions holomorphic in the unit disc U. A connected open set $\mathcal{O} \subseteq \mathbf{C}$ is called an X domain if every holomorphic function defined on U and taking values in $\mathcal{O}$ necessarily belongs to X. Given X one can try to give a "geometrical" characterization of all X domains. For example, if N is the class of functions of bounded characteristic, then $\mathcal{O}$ is an N domain if and only if $\partial\mathcal{O}$, the boundary of $\mathcal{O}$, has positive logarithmic capacity. This is a well-known theorem of Frostman **[2]** and, also, Nevanlinna **[6**, p. 209]. The BMO domains have been characterized by Hayman and Pommerenke **[4]** and also by Stegenga **[8]**. For a summary of some of the results on H^p domains see **[3]**; a more recent result is due to Essén **[1]**. In this note we consider the cases $X = N^+$, the Smirnov class and $X = \Omega$, the class of all outer functions in N^+. We show that $\mathcal{O}$ is an N^+ domain if and only if ∞ is a regular point for the solution of the Dirichlet problem in $\mathcal{O}$, and we show that $\mathcal{O}$ is an Ω domain if and only if both 0 and ∞ are regular points for the solution of the Dirichlet problem in $\mathcal{O}$. In the process we give a characterization of N^- domains; N^- is defined below.

We will use **[7**, Chapter 17] as a reference for the definitions and properties of the classes N, N^+ and H^p, $0 < p \leqslant \infty$. We also use standard results from potential theory (see **[5, 6** and **9]**). When we say that a is regular for $\mathcal{O}$, we mean that $a \notin \mathcal{O}$ and either (i) $a \notin \overline{\mathcal{O}}$ or (ii) $a \in \partial\mathcal{O}$ and is regular for the Dirichlet problem for $\mathcal{O}$.

In each of the cases $X = N^+$, Ω and N^- (defined below), $\mathcal{O}$ will be an X domain if and only if the covering map F: $U \to \mathcal{O}$ is in X. In each case this will follow from subordination and the fact that if G is an outer function then $G \circ b$ is an outer function if $b \in H^\infty$, $b(U) \subseteq U$.

We will assume from now on that $\partial\mathcal{O}$ has positive capacity.

We begin with a special case.

LEMMA. *Suppose that $\mathcal{O}$ is a bounded Ω domain. Then* 0 *is regular for* $\mathcal{O}$.

PROOF. Let F: $U \to \mathcal{O}$ be the covering map; by assumption, $F \in \Omega$. For $z \in \mathcal{O}$, let μ_z be harmonic measure on $\partial\mathcal{O}$ for z. By the principle of the invariance of harmonic

Received by the editors October 6, 1982.
1980 *Mathematics Subject Classification.* Primary 30D50; Secondary 31A15.

measure we have

$$\int_{-\pi}^{\pi} \log|F(e^{i\theta})| P_w(\theta)\, \frac{d\theta}{2\pi} = \int \log|\zeta| d\mu_{F(w)}(\zeta),$$

where P_w is the Poisson kernel for evaluation at $w \in U$. Since $F \in \Omega$ we see that $\log|F(w)| = \int \log|\zeta|\, d\mu_{F(w)}(\zeta)$ and, hence, that $\log|z| = \int \log|\zeta|\, d\mu_z(\zeta)$, since $F(U) = \mathcal{O}$. Take $z, a \in \mathcal{O}$; since $\mathcal{O}$ is bounded we can express Green's function as follows:

$$g(z, a) = \log \frac{1}{|z-a|} - \int \log \frac{1}{|\zeta - a|} d\mu_z(\zeta).$$

Hence,

$$\overline{\lim_{a \to 0}}\, g(z, a) = \log \frac{1}{|z|} - \underline{\lim_{a \to 0}} \int \log \frac{1}{|\zeta - a|} d\mu_z(\zeta).$$

Since $\mathcal{O}$ is bounded, $\log(1/|\zeta - a|)$ is bounded below, and we may apply Fatou's lemma to conclude that

$$\overline{\lim_{a \to 0}}\, g(z, a) \leq \log \frac{1}{|z|} - \int \log \frac{1}{|\zeta|} d\mu_z(\zeta),$$

which is zero by the first part of the proof. This establishes that 0 is regular for $\mathcal{O}$.

In what follows we let N^- denote the set of all $f \in N$ of the form $f = G/\varphi$, where G is an outer function and φ is a nonvanishing inner function.

THEOREM 1. *$\mathcal{O}$ is an N^- domain if and only if 0 is regular for $\mathcal{O}$.*

PROOF. First we suppose that $\mathcal{O}$ is an N^- domain. We consider two cases:

(i) Every circle centered at 0 intersects the complement of $\mathcal{O}$.

(ii) There is a circle centered at 0 which is contained in $\mathcal{O}$.

If (i) holds then 0 is regular for $\mathcal{O}$ by Theorem 10.14 of [5]. Assume (ii) holds, say $\{z: |z| = r\} \subseteq \mathcal{O}$. Let $\mathcal{O}_1 = \{z \in \mathcal{O}: |z| < r\}$. It follows from (ii) that $\mathcal{O}_1$ is connected. Let $f: U \to \mathcal{O}_1$ be holomorphic. Of course, $f: U \to \mathcal{O}$ and, hence, $f \in N^-$. Since f is bounded, $f \in \Omega$; i.e. $\mathcal{O}_1$ is a bounded Ω domain. It follows from the Lemma that 0 is regular for $\mathcal{O}_1$ and, hence, for $\mathcal{O}$, since regularity is a local property.

Now suppose that 0 is regular for $\mathcal{O}$. Let $\mathcal{O}' = \{z: |z| > 1\} \cup \{z: \operatorname{Re} z > 0\}$ and $\mathcal{O}'' = \mathcal{O} \cup \{z: |z| > 1\} \cup \{z: \operatorname{Re} z < 0\}$. Since 0 is regular for $\mathcal{O}$, it follows from the Wiener criteria that 0 is regular for at least one of the domains $\mathcal{O}'$ and $\mathcal{O}''$. Assume 0 is regular for $\mathcal{O}''$. We will show that $\mathcal{O}''$ is an N^- domain, and this will imply that $\mathcal{O}$ is an N^- domain since $\mathcal{O} \subseteq \mathcal{O}''$. To put it another way, we may assume that $\partial\mathcal{O} \subseteq \{z: |z| \leq 1, \operatorname{Re} z > 0\}$ and 0 is regular for $\mathcal{O}$. Now let $\mathcal{O}_n = \{z \in \mathcal{O}: |z| < n\}$. Let $g_n(z, a)$ denote Green's function for $\mathcal{O}_n$, and let μ_z^n denote harmonic measure on $\partial\mathcal{O}_n$ for $z \in \mathcal{O}_n$. Let $g(z, a)$ denote Green's function for $\mathcal{O}$, and let μ_z denote the harmonic measure on $\partial\mathcal{O}$ for z. Then $g_n(z, a) \to g(z, a)$ and $\mu_z^n \to \mu_z$, the measures converging in the weak* sense. Take $a, b \in \mathcal{O}$. Then

$$g_n(z, a) - g_n(z, b) = \log\left|\frac{z-b}{z-a}\right| - \int \log\left|\frac{\zeta - b}{\zeta - a}\right| d\mu_z^n(\zeta).$$

Since $\log|(\zeta - b)/(\zeta - a)|$ is continuous at ∞ and $\mu_z^n \to \mu_z$ weak*, we may let $n \to \infty$ and obtain

$$\begin{aligned} g(z, a) - g(z, b) &= \log\left|\frac{z-b}{z-a}\right| - \int \log\left|\frac{\zeta - b}{\zeta - a}\right| d\mu_z(\zeta) \\ &= \log\frac{1}{|z-a|} - \int \log\frac{1}{|\zeta - a|} d\mu_z(\zeta) - \log\frac{1}{|z-b|} + \int \log\frac{1}{|\zeta - b|} d\mu_z(\zeta). \end{aligned}$$

Now we let $b \to 0$ along the negative real axis. If b is negative and $\zeta \in \partial\mathcal{O}$, then $\log(1/|\zeta - b|) \leq \log(1/|\zeta|)$, so we may apply the dominated convergence theorem and take the limit inside the integral sign. Recalling that 0 is regular and, hence, that $g(z, b) \to 0$ as $b \to 0$; we obtain

$$\begin{aligned} g(z, a) &= \log\frac{1}{|z-a|} - \int \log\frac{1}{|\zeta - a|} d\mu_z(\zeta) - \log\frac{1}{|z|} + \int \log\frac{1}{|\zeta|} d\mu_z(\zeta) \\ &= \int \log\left|\frac{\zeta - a}{z - a}\right| d\mu_z(\zeta) - \log\frac{1}{|z|} + \int \log\frac{1}{|\zeta|} d\mu_z(\zeta). \end{aligned}$$

Now let $a \to \infty$ and arrive at

$$0 < g(z, \infty) = \log|z| - \int \log|\zeta| d\mu_z(\zeta).$$

Take $F\colon U \to \mathcal{O}$, F the covering map; by the Frostman-Nevanlinna result, $F \in N$, so $F = \psi G/\varphi$, where G is an outer function and ψ, φ are nonvanishing inner functions. We have

$$\begin{aligned} \log|F(z)| &= \log|G(z)| + \log|(\psi/\varphi)(z)| \\ &= \int_{-\pi}^{\pi} \log|F(e^{i\theta})| P_z(\theta) \frac{d\theta}{2\pi} + \log\left|\frac{\psi}{\varphi}(z)\right| \\ &= \int \log|\zeta| d\mu_{F(z)}(\zeta) + \log\left|\frac{\psi}{\varphi}(z)\right|, \end{aligned}$$

again by the invariance of harmonic measure. But we have just seen that

$$\log|F(z)| - \int \log|\zeta| d\mu_{F(z)}(\zeta) = g(F(z), \infty) > 0,$$

and so $\log|(\psi/\varphi)(z)| > 0$, i.e. $|\varphi| \leq |\psi|$. This means that $\varphi = \eta\psi$ for some nonvanishing inner function η and, hence,

$$F = \psi G/\varphi = \psi G/\eta\psi = G/\eta \in N^-.$$

THEOREM 2. *$\mathcal{O}$ is an N^+ domain if and only if ∞ is regular for $\mathcal{O}$.*

PROOF. If $0 \notin \mathcal{O}$ let $\mathcal{O}_1 = \{\frac{1}{z}\colon z \in \mathcal{O}\}$. Then Theorem 2 follows by applying Theorem 1 to $\mathcal{O}_1$. Suppose $\mathcal{O}$ is an N^+ domain. Let $\mathcal{O}_2 = \{z\colon \frac{1}{z} \in \mathcal{O}\}$. Note that

$0 \notin \mathcal{O}_2$. If g: $U \to \mathcal{O}_2$ then $f = \frac{1}{g}$: $U \to \mathcal{O}$ and, hence, $f \in N^+$, so $g \in N^-$. That is, $\mathcal{O}_2$ is an N^- domain. By Theorem 1, 0 is regular for $\mathcal{O}_2$ and, hence, ∞ is regular for $\mathcal{O}$.

Next, suppose that ∞ is regular for $\mathcal{O}$ and $0 \in \mathcal{O}$. Let f: $U \to \mathcal{O}$, and let φ: $U \to U \setminus f^{-1}(0)$ be the covering map. Then φ is an inner function (see [2]). Define $G = f \circ \varphi$: $U \to \mathcal{O} \setminus \{0\} = \mathcal{O}_3$. Now ∞ is regular for $\mathcal{O}_3$ and $0 \notin \mathcal{O}_3$, so it follows from the first sentence of the proof that $\mathcal{O}_3$ is an N^+ domain. In particular, $G \in N^+$, say $G = \psi F$, where ψ is a nonvanishing inner function and F is an outer function. Write $f = B\eta h/s$, where B is a Blaschke product, η and φ are nonvanishing inner functions, and h is an outer function. We have $\psi F = (B \circ \varphi)(\eta \circ \varphi)(h \circ \varphi)/s \circ \varphi$. Since the composition of two inner functions is an inner function, we see that $F = h \circ \varphi$, and we are left with $\psi = (B \circ \varphi)(\eta \circ \varphi)/s \circ \varphi$. Let $\lambda = B\eta/s$. Then λ is holomorphic in U, $|\lambda| \leq 1$ on $U \setminus f^{-1}(0)$, and, hence, $|\lambda| \leq 1$ in U. Since $\lambda s = B\eta$ and s is never 0, $\lambda = B\tau$ for some $\tau \in H^\infty$. We get $s\tau = \eta$, from which it follows that $f = B\eta h/s = B\tau h \in N^+$. This completes the proof.

As a consequence of Theorems 1 and 2 we have

THEOREM 3. *$\mathcal{O}$ is an Ω domain if and only if* 0 *and* ∞ *are regular for* $\mathcal{O}$.

Finally, we would like to point out that there is no restriction on the range of an outer function other than it not contain 0. To see this let $\mathcal{O}$ be a domain in $\mathbf{C}$ such that $0 \notin \mathcal{O}$. We will show that there is an outer function F in BMO such that $F(U) = \mathcal{O}$. First there is a function $f \in$ BMO such that $f(U) = \mathbf{C}$ (see [**8**, p. 430]). Let V be a component of $f^{-1}(\mathcal{O})$; clearly $f(V) = \mathcal{O}$. Let σ: $U \to V$ be the covering map. Let $g = f \circ \sigma$. Then we have $g(U) = \mathcal{O}$. Now we take a function $b \in H^\infty$ with the following properties; (i) $b(U) = U$ and (ii) for each θ such that $\lim_{r \to 1} b(re^{i\theta})$ exists, we have $\lim_{r \to 1} |b(re^{i\theta})| < 1$. The existence of such a b follows by constructing an appropriate simply connected surface over U and using the uniformization theorem. Finally let $F = f \circ \sigma \circ b = g \circ b$; $F \in$ BMO because BMO is invariant under composition. We need to check that $F \in \Omega$. We factor $g = sh$, where s is a nonvanishing inner function and h is an outer function. Then $F = g \circ b = (s \circ b)(h \circ b)$. We need only verify that $s \circ b$ is outer. If it were not, it would have radial limit equal to zero somewhere, but this is seen to be impossible by the choice of b.

REFERENCES

1. M. Essén, *On analytic functions which are in H^p for some positive p*, Ark. Mat. **19** (1981), 43–51.

2. O. Frostman, *Potentials d'équilibre et capacité des ensembles avec quelques applications à la théorie des fonctions*, Med. Lunds Univ. Mat. Sem. **3** (1935), 1–118.

3. W. Hayman, *Values and growth of functions regular in the unit disc*, Lecture Notes in Math., vol. 599, Springer-Verlag, Berlin and New York, 1977, pp. 68–75.

4. W. Hayman and C. Pommerenke, *On analytic functions of bounded mean oscillation*, Bull. London Math. Soc. **10** (1978), 219–224.

5. L. Helms, *Introduction to potential theory*, Interscience, New York, 1969.

6. R. Nevanlinna, *Eindeutige analytiche Funktionen*, 2nd ed., Springer-Verlag, Berlin and New York, 1953.

7. W. Rudin, *Real and complex analysis*, 2nd ed. , McGraw-Hill, New York, 1974.

8. D. Stegenga, *A geometric condition which implies BMOA*, Proc. Sympos. Pure Math., vol. 35, Amer. Math. Soc., Providence, R. I., 1979.

9. M. Tsuji, *Potential theory in modern function theory*, Maruzen, Tokyo, 1959.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706 (Current address of Patrick Ahern)

Current address (William Cohn): Department of Mathematics, Wayne State University, Detroit, Michigan 48202