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THE KOBAYASHI INDICATRIX AT THE CENTER OF A CIRCULAR DOMAIN

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ABSTRACT. The indicatrix of the Kobayashi infinitesimal metric at the center of a pseudoconvex complete circular domain coincides with this domain. It follows that a nonconvex complete circular domain cannot be biholomorphic to any convex domain. An example shows that a bounded pseudoconvex complete circular domain in $\mathbf{C}^2$ need not be taut.

Let V be a complex Banach space with norm $\| \;\|$. A *complete circular domain* is a nonempty open set $M \subset V$ such that $\lambda M \subset M$ for all $\lambda \in \mathbf{C}$ with $|\lambda| \leq 1$. Hartogs [**5**, p. 76] encountered these domains while expanding analytic functions into series of homogeneous polynomials; Carathéodory [**2**, p. 104] introduced the term "Kreisgebiet" to describe them. For the purposes of this paper, a *semigauge* on V is an upper semicontinuous function p: $V \to [0, \infty)$ such that $p(\lambda v) = |\lambda| p(v)$ for all $\lambda \in \mathbf{C}$ and $v \in V$; p is called a *gauge* if, in addition, $\| \;\| \leq Cp$ for some constant C.

THEOREM 1. *Let V be a complex Banach space. The formulas*

$$M = \{v \in V: p(v) < 1\}, \quad p(v) = \inf\{\lambda > 0: v \in \lambda M\}$$

establish a one-to-one correspondence between the complete circular domains M and the semigauges p on V. Moreover:

(a) *M is bounded if and only if p is a gauge*;

(b) *M is convex if and only if p is a seminorm*;

(c) *M is pseudoconvex if and only if p is plurisubharmonic.*

In case V has finite dimension:

(d) *M is taut* [**10**, p. 199] *if and only if p is a continuous plurisubharmonic gauge.*

PROOF. Standard normed space techniques yield the one-to-one correspondence and properties (a) and (b).

(c) If p is plurisubharmonic, it follows immediately [**8**, p. 42] that M is pseudoconvex. On the other hand, by definition [**8**, p. 41], M is pseudoconvex if and only if the function $-\log \delta$ is plurisubharmonic, where δ: $M \times (V - \{0\}) \to (0, \infty]$ is defined by

$$\delta(z, v) = \inf\{|\lambda| : \lambda \in \mathbf{C}, z + \lambda v \notin M\}.$$

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Now $p(v) = 1/\delta(0, v) = \exp(-\log \delta(0, v))$ for $v \in V - \{0\}$. Hence, if M is pseudoconvex it follows that p is plurisubharmonic on $V - \{0\}$ and, thus, since p is obviously subharmonic on every complex line through 0, p is plurisubharmonic on V.

(d) Recall that M is *taut* [**1**, p. 430] if and only if the family of all holomorphic mappings from the unit disk $D = \{w \in \mathbf{C}\colon |w| < 1\}$ into M is *normal* in the sense that every sequence of such mappings contains a subsequence converging in the compact-open topology or diverging compactly. ($\{f_j\}$ *diverges compactly* if for any compact sets $K \subset D$, $K' \subset M$ there exists an integer j_0 such that $f_j(K) \cap K' = \varnothing$ for all $j \geqslant j_0$.) If p is a continuous plurisubharmonic gauge, then (a) implies that M is bounded, and it follows (for finite-dimensional V) that every sequence of holomorphic mappings from D into M contains a subsequence $\{f_j\}$ converging to a holomorphic mapping $f\colon D \to V$; since $p \circ f_j < 1$ and p is continuous, we have $p \circ f \leqslant 1$; by the maximum principle for subharmonic functions, either $p \circ f < 1$ (so that $f\colon D \to M$), or $p \circ f \equiv 1$ (so that $\{f_j\}$ diverges compactly); thus M is taut. On the other hand, if M is taut then (c) implies that p is plurisubharmonic. To complete the proof it will suffice to show that if M is unbounded or p is discontinuous, then M cannot be taut. Now if M is unbounded there are points $v_j \in M$ with $\|v_j\| \to \infty$; the holomorphic mappings $f_j\colon D \to M$ defined by $f_j(w) = wv_j$ satisfy $f_j(0) = 0$ and $\|f_j(w)\| \to \infty$ for $w \neq 0$, so M is not taut. Finally if p is discontinuous there are points $v_j \to v_0 \in M$ with $p(v_j) \nrightarrow p(v_0)$; since p is upper semicontinuous we may assume that $p(v_j) \leqslant b < p(v_0)$ for all j; the holomorphic mappings $f_j\colon D \to M$ defined by $f_j(w) = wv_j/b$ satisfy $f_j(0) = 0$, while $f_j(b/p(v_0)) = v_j/p(v_0) \to v/p(v_0) \notin M$; thus M is not taut. □

Among the more tractable complete circular domains are the complete Reinhardt domains in $\mathbf{C}^n$. Recently P. Pflug [**9**] has shown that every bounded pseudoconvex complete Reinhardt domain is finitely compact with respect to its Carathéodory distance; hence such a domain is taut. Nevertheless, by modifying an example due to N. Kerzman [**6**, pp. 180–181], we can construct a bounded pseudoconvex complete circular domain in $\mathbf{C}^2$ that is not taut. Indeed, by Theorem 1 it suffices to observe that the formula

$$p(z, w) = \exp\left(\max\left(\log|z|,\, 1 + \sum_{n=1}^{\infty} 2^{-n} \log|nw - z|\right)\right)$$

defines a plurisubharmonic gauge $p\colon \mathbf{C}^2 \to [0, \infty)$ that is not continuous at the point $z = 1$, $w = 0$.

Let $D = \{w \in \mathbf{C}\colon |w| < 1\}$ be the open unit disk, and let $\mathrm{Hol}(X, Y)$ denote the set of holomorphic mappings from X into Y. The *Carathéodory* and *Kobayashi differential metrics* at the point z of the complex (Banach) manifold M are the semigauges on the complex tangent space $T_z(M)$ defined by

$$E_M(v) = \sup\left\{|g_*(v)| : g \in \mathrm{Hol}(M, D) \text{ and } g(z) = 0\right\}$$

and

$$F_M(v) = \inf\left\{|u| : u \in T_0(D) \text{ and } f_*(u) = v \text{ for some } f \in \mathrm{Hol}(D, M)\right\}$$

[7, pp. 360–361]. Here we have identified $T_0(D)$ with $\mathbf{C}$ so that $|\ |$ denotes the ordinary absolute value. The corresponding complete circular domains

$$\Gamma_z(M) = \{v \in T_z(M) \colon E_M(v) < 1\} \text{ and } \Delta_z(M) = \{v \in T_z(M) \colon F_M(v) < 1\}$$

are called the *indicatrices* of these metrics at z [7, p. 399]. In case M is an open subset of the Banach space V and $z = 0$, we can identify $T_z(M)$ with V [3, pp. 113–114].

THEOREM 2. *Let M be a complete circular domain. Then $\Gamma_0(M) \supset \Delta_0(M) \supset M$.*
(a) [2, *Satz* 5, *p.* 120] *If M is convex, then $\Gamma_0(M) = M$.*
(b) *If M is pseudoconvex, then $\Delta_0(M) = M$.*

PROOF. The inclusion $\Gamma_z(M) \supset \Delta_z(M)$ holds at any point z in a complex manifold. Now let $p\colon V \to [0, \infty)$ be the semigauge associated with the complete circular domain M by Theorem 1. Let $v \in V$; for each $u > p(v)$ the formula $f(w) = wv/u$ defines a mapping $f \in \mathrm{Hol}(D, M)$ with $f_*(u) = v$; thus $F_M(v) \leqslant p(v)$. Therefore $\Delta_0(M) \supset M$.

(a) [3, Lemma V.1.5, p. 116]. Assume that M is convex. By Theorem 1(b), p is a continuous seminorm. Let $v \in V$; the Hahn-Banach theorem gives a continuous linear functional $h\colon V \to \mathbf{C}$ such that $|h| \leqslant p$ and $h(v) = p(v)$; now $g = h|_M \in \mathrm{Hol}(M, D)$, $g(0) = 0$, and $|g_*(v)| = |h(v)| = p(v)$; thus $E_M(v) \geqslant p(v)$.

(b) Assume that M is pseudoconvex. By Theorem 1(c), p is plurisubharmonic. Let $v \in \Delta_0(M)$. Then there exist $u \in T_0(M)$ and $f \in \mathrm{Hol}(D, M)$ such that $f(0) = 0$, $uf'(0) = v$, and $F_M(v) \leqslant |u| < 1$. Expanding f in a power series, we see that the association $w \to f(w)/w$ extends to a holomorphic mapping $g\colon D \to V$ with $g(0) = f'(0)$. Take $0 < R < 1$. For $|w| = R$ we have $p \circ g(w) = p(f(w)/w) = p(f(w))/R < 1/R$. Since $p \circ g$ is subharmonic, $p \circ g(0) < 1/R$. Letting $R \to 1$ we get

$$p(v) = p(uf'(0)) = |u|\, p(f'(0)) = |u|\, p \circ g(0) \leqslant |u| < 1,$$

i.e., $v \in M$. $\square$

COROLLARY. *If the complete circular domain M is biholomorphic to a convex domain, then M is convex.*

PROOF. A biholomorphic mapping $f\colon M \to N$ induces a linear biholomorphic mapping $f_*\colon \Delta_0(M) \to \Delta_z(N)$ where $z = f(0)$ [7, p. 399]. If N is convex, then $\Delta_z(N)$ is convex [4, Exercise 13, pp. 397–398], hence $\Delta_0(M)$ is convex; also N is pseudoconvex and, since pseudoconvexity is biholomorphic invariant, so is M; by Theorem 2(b), $M = \Delta_0(M)$ is convex. $\square$

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