THE KOBAYASHI INDICATRIX
AT THE CENTER OF A CIRCULAR DOMAIN

THEODORE J. BARTH

Abstract. The indicatrix of the Kobayashi infinitesimal metric at the center of a pseudoconvex complete circular domain coincides with this domain. It follows that a nonconvex complete circular domain cannot be biholomorphic to any convex domain. An example shows that a bounded pseudoconvex complete circular domain in $\mathbb{C}^2$ need not be taut.

Let $V$ be a complex Banach space with norm $\| \|$. A complete circular domain is a nonempty open set $M \subset V$ such that $\lambda M \subset M$ for all $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$. Hartogs [5, p. 76] encountered these domains while expanding analytic functions into series of homogeneous polynomials; Carathéodory [2, p. 104] introduced the term “Kreisgebiet” to describe them. For the purposes of this paper, a semigauge on $V$ is an upper semicontinuous function $p: V \to [0, \infty)$ such that $p(\lambda v) = |\lambda| p(v)$ for all $\lambda \in \mathbb{C}$ and $v \in V$; $p$ is called a gauge if, in addition, $\| \| \leq Cp$ for some constant $C$.

Theorem 1. Let $V$ be a complex Banach space. The formulas

$$M = \{ v \in V : p(v) < 1 \}, \quad p(v) = \inf \{ \lambda > 0 : \lambda M \}$$

establish a one-to-one correspondence between the complete circular domains $M$ and the semigauges $p$ on $V$. Moreover:

(a) $M$ is bounded if and only if $p$ is a gauge;
(b) $M$ is convex if and only if $p$ is a seminorm;
(c) $M$ is pseudoconvex if and only if $p$ is plurisubharmonic.

In case $V$ has finite dimension:
(d) $M$ is taut [10, p. 199] if and only if $p$ is a continuous plurisubharmonic gauge.

Proof. Standard normed space techniques yield the one-to-one correspondence and properties (a) and (b).

(c) If $p$ is plurisubharmonic, it follows immediately [8, p. 42] that $M$ is pseudoconvex. On the other hand, by definition [8, p. 41], $M$ is pseudoconvex if and only if the function $-\log \delta$ is plurisubharmonic, where $\delta: M \times (V - \{0\}) \to (0, \infty)$ is defined by

$$\delta(z, v) = \inf \{ |\lambda| : \lambda \in \mathbb{C}, z + \lambda v \notin M \}.$$
Now $p(v) = 1/\delta(0, v) = \exp(-\log \delta(0, v))$ for $v \in V - \{0\}$. Hence, if $M$ is pseudo-convex it follows that $p$ is plurisubharmonic on $V - \{0\}$ and, thus, since $p$ is obviously subharmonic on every complex line through 0, $p$ is plurisubharmonic on $V$.

(d) Recall that $M$ is taut [1, p. 430] if and only if the family of all holomorphic mappings from the unit disk $D = \{w \in \mathbb{C}: |w| < 1\}$ into $M$ is normal in the sense that every sequence of such mappings contains a subsequence converging in the compact-open topology or diverging compactly. \(\{f_j\}\) diverges compactly if for any compact sets $K \subset D$, $K' \subset M$ there exists an integer $j_0$ such that $f_j(K) \cap K' = \emptyset$ for all $j > j_0$. If $p$ is a continuous plurisubharmonic gauge, then (a) implies that $M$ is bounded, and it follows (for finite-dimensional $V$) that every sequence of holomorphic mappings from $D$ into $M$ contains a subsequence \(\{f_j\}\) converging to a holomorphic mapping $f: D \to V$; hence $p \circ f < 1$ and $p$ is continuous, we have $p \circ f \leq 1$; by the maximum principle for subharmonic functions, either $p \circ f < 1$ (so that $f: D \to M$), or $p \circ f \equiv 1$ (so that \(\{f_j\}\) diverges compactly); thus $M$ is taut. On the other hand, if $M$ is taut then (c) implies that $p$ is plurisubharmonic. To complete the proof it will suffice to show that if $M$ is unbounded or $p$ is discontinuous, then $M$ cannot be taut. Now if $M$ is unbounded there are points $v_j \in M$ with $\|v_j\| \to \infty$; the holomorphic mappings $f_j: D \to M$ defined by $f_j(w) = wv_j$ satisfy $f_j(0) = 0$ and $\|f_j(w)\| \to \infty$ for $w \neq 0$, so $M$ is not taut. Finally if $p$ is discontinuous there are points $v_j \to v_0 \in M$ with $p(v_j) \not\to p(v_0)$; since $p$ is upper semicontinuous we may assume that $p(v_j) \leq b < p(v_0)$ for all $j$; the holomorphic mappings $f_j: D \to M$ defined by $f_j(w) = wv_j/b$ satisfy $f_j(0) = 0$, while $f_j(b/p(v_0)) = v_j/p(v_0) \to v/p(v_0) \not\in M$; thus $M$ is not taut. \(\square\)

Among the more tractable complete circular domains are the complete Reinhardt domains in $\mathbb{C}^n$. Recently P. Pflug [9] has shown that every bounded pseudoconvex complete Reinhardt domain is finitely compact with respect to its Carathéodory distance; hence such a domain is taut. Nevertheless, by modifying an example due to N. Kerzman [6, pp. 180–181], we can construct a bounded pseudoconvex complete circular domain in $\mathbb{C}^2$ that is not taut. Indeed, by Theorem 1 it suffices to observe that the formula

$$p(z, w) = \exp \left( \max \left( \log |z| , 1 + \sum_{n=1}^{\infty} 2^{-n} \log |nw - z| \right) \right)$$

defines a plurisubharmonic gauge $p: \mathbb{C}^2 \to [0, \infty)$ that is not continuous at the point $z = 1, w = 0$.

Let $D = \{w \in \mathbb{C}: |w| < 1\}$ be the open unit disk, and let Hol($X, Y$) denote the set of holomorphic mappings from $X$ into $Y$. The Carathéodory and Kobayashi differential metrics at the point $z$ of the complex (Banach) manifold $M$ are the semigauges on the complex tangent space $T_z(M)$ defined by

$$E_M(v) = \sup \{|g_*(v)| : g \in \text{Hol}(M, D) \text{ and } g(z) = 0\}$$

and

$$F_M(v) = \inf \{|u| : u \in T_0(D) \text{ and } f_*(u) = v \text{ for some } f \in \text{Hol}(D, M)\}$$
Here we have identified \( T_0(D) \) with \( C \) so that \(| |\) denotes the ordinary absolute value. The corresponding complete circular domains

\[
\Gamma_z(M) = \{ v \in T_z(M) : E_M(v) < 1 \}
\]

and

\[
\Delta_z(M) = \{ v \in T_z(M) : F_M(v) < 1 \}
\]

are called the indicatrices of these metrics at \( z \) [7, p. 399]. In case \( M \) is an open subset of the Banach space \( V \) and \( z = 0 \), we can identify \( T_z(M) \) with \( V \) [3, pp. 113–114].

**Theorem 2.** Let \( M \) be a complete circular domain. Then \( \Gamma_0(M) \supset \Delta_0(M) \supset M \).

(a) \([2, \text{ Satz 5, p. 120}]\) If \( M \) is convex, then \( \Gamma_0(M) = M \).

(b) If \( M \) is pseudoconvex, then \( \Delta_0(M) = M \).

**Proof.** The inclusion \( \Gamma_z(M) \supset \Delta_z(M) \) holds at any point \( z \) in a complex manifold. Now let \( p : V \to [0, \infty) \) be the semigauge associated with the complete circular domain \( M \) by Theorem 1. Let \( v \in V \); for each \( u > p(v) \) the formula \( f(w) = wv/u \) defines a mapping \( f \in \text{Hol}(D, M) \) with \( f_* = v \); thus \( F_M(v) \leq p(v) \). Therefore \( \Delta_0(M) \supset M \).

(a) \([3, \text{ Lemma V.1.5, p. 116}]\). Assume that \( M \) is convex. By Theorem 1(b), \( p \) is a continuous seminorm. Let \( v \in V \); the Hahn-Banach theorem gives a continuous linear functional \( h : V \to \mathbb{C} \) such that \( |h| \leq p \) and \( h(v) = p(v) \); now \( g = h |_M \in \text{Hol}(M, D), g(0) = 0, \) and \( |g_*| = |h(v)| = p(v) \); thus \( E_M(v) \geq p(v) \).

(b) Assume that \( M \) is pseudoconvex. By Theorem 1(c), \( p \) is plurisubharmonic. Let \( v \in \Delta_0(M) \). Then there exist \( u \in T_0(M) \) and \( f \in \text{Hol}(D, M) \) such that \( f(0) = 0, u f'(0) = v \), and \( F_M(v) \leq |u| < 1 \). Expanding \( f \) in a power series, we see that the association \( w \to f(w)/w \) extends to a holomorphic mapping \( g : D \to V \) with \( g(0) = f'(0) \). Take \( 0 < R < 1 \). For \( |w| = R \) we have \( p \circ g(w) = p(f(w)/w) = p(f(w))/R < 1/R \). Since \( p \circ g \) is subharmonic, \( p \circ g(0) < 1/R \). Letting \( R \to 1 \) we get

\[
p(v) = p(u f'(0)) = |u| p(f'(0)) = |u| p \circ g(0) \leq |u| < 1,
\]

i.e., \( v \in M \). □

**Corollary.** If the complete circular domain \( M \) is biholomorphic to a convex domain, then \( M \) is convex.

**Proof.** A biholomorphic mapping \( f : M \to N \) induces a linear biholomorphic mapping \( f_* : \Delta_0(M) \to \Delta_0(N) \) where \( z = f(0) \) [7, p. 399]. If \( N \) is convex, then \( \Delta_0(N) \) is convex [4, Exercise 13, pp. 397–398], hence \( \Delta_0(M) \) is convex; also \( N \) is pseudoconvex and, since pseudoconvexity is biholomorphic invariant, so is \( M \); by Theorem 2(b), \( M = \Delta_0(M) \) is convex. □

**References**


