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STRONG CONVERGENCE OF MARTINGALES IN VON NEUMANN ALGEBRAS

MAKOTO TSUKADA

ABSTRACT. In this paper we prove strong and L^1-norm convergence of martingales with respect to a faithful normal semifinite weight on a von Neumann algebra.

1. Introduction. Let M be a von Neumann algebra and φ be a faithful normal semifinite weight on M_+ (the positive cone of M). We denote by $\mathfrak{n}_\varphi$ the set of all elements $x \in M$ with $\varphi(x^*x) < +\infty$ and by $\mathfrak{m}_\varphi$ the linear span of $\mathfrak{n}_\varphi^*\mathfrak{n}_\varphi$. Then φ is uniquely extended to a linear functional on $\mathfrak{m}_\varphi$ and we also denote it by φ. For a von Neumann subalgebra N of M, if there exists a faithful σ-weakly continuous projection ε of norm one from M onto N such that $\varphi(x) = \varphi(\varepsilon(x))$ for every $x \in \mathfrak{m}_\varphi$, we call ε the conditional expectation onto N with respect to φ. Takesaki [6] proved that there exists a unique conditional expectation onto N with respect to φ if and only if φ is semifinite on N (i.e., $N \cap \mathfrak{m}_\varphi$ is σ-weakly dense in N) and N is globally invariant under the modular automorphism group associated with φ.

Let M_* be the predual of M (i.e., the set of all σ-weakly continuous linear functionals on M). Suppose that N is a von Neumann subalgebra of M such that the conditional expectation ε onto N with respect to φ exists. Then we define $L^1(N; \varepsilon)$ as the set of all elements $\psi \in M_*$ with $\psi = \psi \circ \varepsilon$. For any $\psi \in M_*$ we define $\varepsilon^*(\psi) = \psi \circ \varepsilon$.

THEOREM 1. *$L^1(N; \varepsilon)$ is isometrically isomorphic to the predual of N. Moreover ε^* is a projection of norm one from M_* onto $L^1(N; \varepsilon)$.*

PROOF. Let N_* be the predual of N. For any $\psi \in N_*$ we define $\iota(\psi) = \psi \circ \varepsilon$. Then $\iota(\psi) \in L^1(N; \varepsilon)$ and

$$\|\psi\| = \sup_{\substack{x \in N \\ \|x\| \leq 1}} |\psi(x)| = \sup_{\substack{x \in M \\ \|x\| \leq 1}} |\psi(\varepsilon(x))| = \|\iota(\psi)\|,$$

because ε is a projection of norm one from M onto N. Moreover $\iota(\psi \restriction_N) = \psi$ for every $\psi \in L^1(N; \varepsilon)$. Hence ι is an isometrical isomorphism from N_* onto $L^1(N; \varepsilon)$. On the other hand for any $\psi \in M_*$ it follows that

$$\|\varepsilon^*(\psi)\| = \sup_{\substack{x \in M \\ \|x\| \leq 1}} |\psi(\varepsilon(x))| \leq \sup_{\substack{x \in M \\ \|x\| \leq 1}} |\psi(x)| = \|\psi\|.$$

Thus ε^* is of norm one. It is clear that ε^* is a projection from M_* onto $L^1(N; \varepsilon)$.

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Suppose that φ is a state. Goldstein [4] embedded M into M_* and defined $L^1(N)$ as the closure of N in M_* for any von Neumann subalgebra N of M. If the conditional expectation ε onto N with respect to φ exists, $L^1(N)$ coincides with $L^1(N; \varepsilon)$ defined as above. In that paper he proved the strong convergence of martingales. We generalize this to the case in which φ is a weight.

2. Increasing martingales. In this section we fix an increasing net $\{N_\alpha\}$ of von Neumann subalgebras of M such that the conditional expectation ε_α onto N_α with respect to φ exists for every α.

THEOREM 2. *The conditional expectation ε_∞ onto $\bigvee_\alpha N_\alpha$ with respect to φ exists and the following assertions are satisfied*:

(i) $\{\varepsilon_\alpha(x)\}$ *converges to* $\varepsilon_\infty(x)$ *in the s^*-topology for every* $x \in M$;

(ii) $\{\varepsilon_\alpha^*(\psi)\}$ *converges to* $\varepsilon_\infty^*(\psi)$ *in the norm for every* $\psi \in M_*$.

PROOF. Since φ is semifinite on N_α and N_α is globally invariant under the modular automorphism group associated with φ for every α, φ is semifinite on $\bigvee_\alpha N_\alpha$ and $\bigvee_\alpha N_\alpha$ is globally invariant under the modular automorphism group. Hence there exists a unique conditional expectation ε_∞ onto $\bigvee_\alpha N_\alpha$ with respect to φ. By the uniqueness of the conditional expectation $\varepsilon_\alpha = \varepsilon_\alpha \circ \varepsilon_\infty$ for every α. Therefore to prove (i) and (ii) we may assume that $M = \bigvee_\alpha N_\alpha$ without loss of generality.

(i) We first prove that if $\varepsilon_\alpha(x) = 0$ for every α, then $x = 0$. Assume that $\varepsilon_\alpha(x) = 0$ for every α. We fix an index α_0 and for any $\psi \in M_*$ we put $\hat{\psi} = \psi \circ \varepsilon_{\alpha_0}$. Then for any $\alpha \geqslant \alpha_0$ and $a \in N_\alpha$ we have $\hat{\psi}(ax) = \hat{\psi}(\varepsilon_\alpha(ax)) = \hat{\psi}(a\varepsilon_\alpha(x)) = 0$. Therefore $\hat{\psi}(ax) = 0$ for any $a \in \bigcup_\alpha N_\alpha$. Since $\bigcup_\alpha N_\alpha$ is σ-weakly dense in M, we have $\hat{\psi}(ax) = 0$ for any $a \in M$. Hence $\psi(\varepsilon_{\alpha_0}(x^*x)) = \hat{\psi}(x^*x) = 0$. Since $\psi \in M_*$ is arbitrary, $\varepsilon_{\alpha_0}(x^*x) = 0$, and since ε_{α_0} is faithful, it follows that $x = 0$. Next we prove that $\{\varepsilon_\alpha(x)\}$ converges to x in the σ-weak topology for any $x \in M$. Since $\{\varepsilon_\alpha(x)\}$ is uniformly bounded, for any subnet $\{\varepsilon_{\alpha'}(x)\}$ there exists a subnet $\{\varepsilon_{\alpha''}(x)\}$ which converges to some $y \in M$ in the σ-weak topology. For every α, because of σ-weak continuity of ε_α, it follows that $\varepsilon_\alpha(\varepsilon_{\alpha''}(x))$ tends to $\varepsilon_\alpha(y)$ as $\alpha'' \uparrow$ in the σ-weak topology. Here for sufficiently large α'' we have $\varepsilon_\alpha(\varepsilon_{\alpha''}(x)) = \varepsilon_\alpha(x)$. Hence $\varepsilon_\alpha(x) = \varepsilon_\alpha(y)$ for every α and we have $x = y$ by the fact that is proved above. Thus $\{\varepsilon_\alpha(x)\}$ converges to x in the σ-weak topology. Moreover for any $\psi \in M_*^+$ it follows that

$$\begin{aligned}
0 &\leqslant \psi((\varepsilon_\alpha(x) - x)^*(\varepsilon_\alpha(x) - x)) \\
&= \psi(\varepsilon_\alpha(x)^*\varepsilon_\alpha(x)) - \psi(\varepsilon_\alpha(x)^*x) - \psi(x^*\varepsilon_\alpha(x)) + \psi(x^*x) \\
&\leqslant \psi(\varepsilon_\alpha(x^*x)) - \psi(\varepsilon_\alpha(x)^*x) - \psi(x^*\varepsilon_\alpha(x)) + \psi(x^*x) \\
&\to 0 \quad (\text{as } \alpha \uparrow).
\end{aligned}$$

$\psi((\varepsilon_\alpha(x) - x)(\varepsilon_\alpha(x) - x)^*)$ tends to 0 as $\alpha \uparrow$ in the same way and we have (i).

(ii) Since $\{\varepsilon_\alpha(x)\}$ converges to x in the σ-weak topology for any $x \in M$, $\{\varepsilon_\alpha^*(\psi)\}$ converges to ψ in the weak topology for any $\psi \in M_*$. Therefore $\bigcup_\alpha L^1(N_\alpha; \varepsilon_\alpha)$ is weakly dense in M_* and so in the norm. Hence we have (ii) by the standard argument.

REMARK. The strong convergence of martingales was proved by Connes [2] for the case in which M is σ-finite, and by Lance [5] for the case in which $\bigvee_\alpha N_\alpha$ is semifinite.

3. Decreasing martingales. In this section we fix a decreasing net $\{N_\alpha\}$ of von Neumann subalgebras of M such that the conditional expectation ε_α onto N_α with respect to φ exists for every α.

THEOREM 3. *Suppose that φ is semifinite on $\bigcap_\alpha N_\alpha$. Then the conditional expectation ε_∞ onto $\bigcap_\alpha N_\alpha$ with respect to φ exists and the following assertions are satisfied*:

(i) $\{\varepsilon_\alpha(x)\}$ *converges to $\varepsilon_\infty(x)$ in the s^*-topology for every $x \in M$*;

(ii) $\{\varepsilon_\alpha^*(\psi)\}$ *converges to $\varepsilon_\infty^*(\psi)$ in the norm for every $\psi \in M_*$.*

PROOF. Since N_α is globally invariant under the modular automorphism group associated with φ for every α, so is $\bigcap_\alpha N_\alpha$, and since φ is semifinite on $\bigcap_\alpha N_\alpha$, there exists a unique conditional expectation ε_∞ onto $\bigcap_\alpha N_\alpha$ with respect to φ. By the uniqueness of the conditional expectation, $\varepsilon_\infty = \varepsilon_\infty \circ \varepsilon_\alpha$ for every α.

(ii) Let any $\psi \in M_*$ be fixed. Then

$$\begin{aligned}\|\varepsilon_\alpha^*(\psi) - \varepsilon_\infty^*(\psi)\| &= \sup_{\substack{x \in M \\ \|x\| \leq 1}} |\psi(\varepsilon_\alpha(x)) - \psi(\varepsilon_\infty(x))| \\ &= \sup_{\substack{x \in M \\ \|x\| \leq 1}} |\psi(\varepsilon_\alpha(x)) - \psi(\varepsilon_\infty(\varepsilon_\alpha(x)))| \\ &= \sup_{\substack{x \in N_\alpha \\ \|x\| \leq 1}} |\psi(x) - \psi(\varepsilon_\infty(x))| = |\psi(x_\alpha) - \psi(\varepsilon_\infty(x_\alpha))|\end{aligned}$$

for some $x_\alpha \in N_\alpha$ with $\|x_\alpha\| \leq 1$. Since $\{x_\alpha\}$ is uniformly bounded, for any subnet $\{x_{\alpha'}\}$ there exists a subnet $\{x_{\alpha''}\}$ which converges to some $x_\infty \in M$ in the σ-weak topology. Then $x_\infty \in \bigcap_\alpha N_\alpha$ and $|\psi(x_{\alpha''}) - \psi(\varepsilon_\infty(x_{\alpha''}))|$ tends to 0 as $\alpha'' \uparrow$. Therefore it follows that $\{\varepsilon_\alpha^*(\psi)\}$ converges to $\varepsilon_\infty^*(\psi)$ in the norm.

(i) It can be easily seen that $\{\varepsilon_\alpha(x)\}$ converges to $\varepsilon_\infty(x)$ in the σ-weak topology in the same way as the proof of Theorem 2(i). For any $\psi \in M_*^+$ we have

$$\begin{aligned}\psi((\varepsilon_\alpha(x) - \varepsilon_\infty(x))^*(\varepsilon_\alpha(x) - \varepsilon_\infty(x))) &= \psi(\varepsilon_\alpha(x)^*\varepsilon_\alpha(x)) - \psi(\varepsilon_\alpha(x)^*\varepsilon_\infty(x)) \\ &\quad - \psi(\varepsilon_\infty(x)^*\varepsilon_\alpha(x)) + \psi(\varepsilon_\infty(x)^*\varepsilon_\infty(x)).\end{aligned}$$

Here the second and third terms of the right-hand side tend to $\psi(\varepsilon_\infty(x)^*\varepsilon_\infty(x))$ as $\alpha \uparrow$. Moreover the first term is equal to $\varepsilon_\alpha^*(\psi)(\varepsilon_\alpha(x)^*x)$. Here $\varepsilon_\alpha^*(\psi)$ tends to $\varepsilon_\infty^*(\psi)$ as $\alpha \uparrow$ in the norm and $\{\varepsilon_\alpha(x)\}$ is uniformly bounded and converges to $\varepsilon_\infty(x)$ in σ-weak topology. So it follows that $\varepsilon_\alpha^*(\psi)(\varepsilon_\alpha(x)^*x)$ tends to $\varepsilon_\infty^*(\psi)(\varepsilon_\infty(x)^*x) = \psi(\varepsilon_\infty(x)^*\varepsilon_\infty(x))$ as $\alpha \uparrow$. Thus $\psi((\varepsilon_\alpha(x) - \varepsilon_\infty(x))^*(\varepsilon_\alpha(x) - \varepsilon_\infty(x)))$ tends to 0 as $\alpha \uparrow$. $\psi((\varepsilon_\alpha(x) - \varepsilon_\infty(x))(\varepsilon_\alpha(x) - \varepsilon_\infty(x))^*)$ tends to 0 as $\alpha \uparrow$ in the same way.

In the above theorem the condition that φ is semifinite on $\bigcap_\alpha N_\alpha$ is necessary. Let $M = l^\infty$ and for $\{x_i\} \in M_+$ define $\varphi(\{x_i\}) = \sum_{i=1}^\infty x_i$. Then M is a von Neumann algebra and φ is a faithful normal semifinite weight on M_+. Now let N_n be the set of all elements $\{x_i\} \in M$ such that $x_1 = x_2 = \cdots = x_n$. Then $\{N_n\}$ is a decreasing sequence of von Neumann subalgebras of M. We can easily see that the conditional

expectation ε_n onto N_n with respect to φ exists and for any $\{x_i\} \in M$, $\varepsilon_n(\{x_i\})$ is the sequence $\{y_i\}$ such that $y_i = (1/n) \cdot \Sigma_{k=1}^n x_k$ for $i = 1, \ldots, n$ and $y_i = x_i$ for $i = n + 1, \ldots$. Then $\bigcap_n N_n = \{\lambda \cdot I\colon \lambda \in \mathbf{C}\}$, on which φ is not semifinite. Moreover $\{\varepsilon_n(\{x_i\})\}$ does not always converge in the σ-weak topology, because $\{(1/n) \cdot \Sigma_{k=1}^n x_k\}$ does not always converge.

In the increasing case (resp. the decreasing case) a sequence $\{x_\alpha\}$ is called a *martingale* if $x_\alpha \in N_\alpha$ for every α and $\varepsilon_\alpha(x_\beta) = x_\alpha$ whenever $\alpha \preccurlyeq \beta$ (resp. $\alpha \succcurlyeq \beta$). The sequence $\{\varepsilon_\alpha(x)\}$ in Theorem 2 (resp. Theorem 3) is an example of martingales and such a martingale is called *simple*. In the decreasing case any martingale $\{x_\alpha\}$ is essentially simple by considering $\{x_\alpha\}_{\alpha \succcurlyeq \alpha_0}$ for any fixed α_0. In the increasing case, examining the proof of Theorem 2, we can easily see that a martingale is simple if and only if it is uniformly bounded. Similarly, L^1-martingales in M_* can be considered and it is seen that an increasing L^1-martingale is simple if it is relatively weakly compact in M_*.

Recently Accardi and Cecchini [**1**] generalized the notion of conditional expectations on von Neumann algebras. In our forthcoming paper [**8**] we investigate strong convergence of martingales in terms of the generalized conditional expectation.

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Department of Information Sciences, Science University of Tokyo, Noda City, Chiba 278, Japan