PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 88, Number 4, August 1983

MINIMAL IDEALS IN QUADRATIC JORDAN ALGEBRAS

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ABSTRACT. In associative and alternative algebras a minimal ideal is either trivial or simple. This is not known for quadratic Jordan algebras. In the present note we show that a minimal ideal is either trivial or $\mathfrak{D}$-simple (possesses no proper ideals invariant under all inner derivations induced from the ambient algebra). In particular, the heart of any quadratic Jordan algebra is either trivial or $\mathfrak{D}$-simple. Hearts have recently played an important role in Zelmanov's theory of prime Jordan algebras.

A *unital quadratic Jordan algebra* over an arbitrary ring of scalars Φ is a Φ-module J with unit element 1 and composition $U_x y$ quadratic in x and linear in y such that the axioms

(0.1) $U_1 = 1$,

(0.2) $U_x V_{y,x} = V_{x,y} U_x = U_{U(x)y,x}$,

(0.3) $U_{U(x)y} = U_x U_y U_x$

hold strictly, where we introduce the notations $V_{x,y}(z) = \{x\, y\, z\} = U_{x,z}(y)$ for $U_{x,z} = U_{x+z} - U_x - U_z$; the square is given by $x^2 = U_x 1$, with linearization $V_x(y) = x \circ y = U_{x,y} 1$. A (nonunital) *quadratic Jordan algebra* J has compositions $U_x y$ and x^2 satisfying certain axioms which guarantee that its *unital hull* $\hat{J} = \Phi 1 + J$ is a unital quadratic Jordan algebra.

A *derivation* of a quadratic Jordan algebra is a linear transformation D such that

$$(0.4) \qquad D(x^2) = x \circ D(x), \qquad D(U_x y) = U_{D(x),x} y + U_x D(y).$$

The *inner derivations* are given by

$$(0.5) \qquad D_{x,y} = [V_x, V_y] = V_{x,y} - V_{y,x}.$$

An *ideal* $I \lhd J$ is a subspace which is both an *outer ideal* ($U_{\hat{J}} I \subset I$, i.e. $U_J I \subset I$ and $V_J I \subset I$) and an *inner ideal* ($U_I \hat{J} \subset I$, i.e. $U_I J \subset I$ and $I^2 \subset I$). An outer ideal I is inner as soon as $U_t \hat{J} \subset I$ for some spanning set $\{t\}$ for I. An algebra is *simple* if it is not trivial and has no proper ideals; it is *$\mathfrak{D}$-simple* if it is not trivial and has no proper ideals which are invariant under all derivations.

A *subideal* K of J is a subspace which is n steps removed (for some n) from being an ideal, $K \lhd I_1 \lhd I_2 \lhd \cdots \lhd I_n \lhd J$. We are interested in how far a subideal is

Received by the editors May 11, 1982.

1980 *Mathematics Subject Classification*. Primary 17C10, 17C20; Secondary 17C30, 17C65.

[1] The author wishes to thank the Department of Pure Mathematics at the University of Sydney for its hospitality and Dr. K. G. Choo for his assistance and encouragement.

[2] Research partially supported by NSF Grant MCS-80-02319.

0002-9939/82/0000-1108/$02.00

from being an ideal. The question of whether a minimal ideal $I \lhd J$ is simple (resp. $\mathfrak{D}$-simple) depends on whether there are any proper (resp. $\mathfrak{D}$-invariant) subideals $K \lhd I \lhd J$; our goal will be to show that each such K would be an ideal in J, therefore by minimality of I either $K = I$ or $K = 0$, so K would be improper.

In general, if $I, K \lhd J$ then their quadratic product $U_I K \lhd J$ also. In particular, if $I \lhd J$ then $I^3 = U_I I \lhd J$ also. We say I is *trivial* if $I^3 = 0$ (this is weaker than requiring $I^2 = 0$) and *idempotent* if $I^3 = I$. If I is a minimal ideal in J, then the J-ideal $I^3 \subset I$ can only be 0 or I, so I is either trivial or idempotent.

1. Subideals. We gather here some technical results on subideals of idempotent ideals. These require some further properties of multiplication operators in Jordan algebras (see [**2**]):

(1.1) $V_x = U_{x,1} = V_{x,1} = V_{1,x}$,

(1.2) $\{x\,x\,z\} = x^2 \circ z, \quad \{xyz\} + \{yxz\} = (x \circ y) \circ z$,

(1.3) $V_{z,z} = V_{z^2}, \quad V_{z,y} = V_z V_y - U_{z,y}, \quad 2U_z = V_z^2 - V_{z^2}$,

(1.4) $V_{U(x)y,y} = V_{x,U(y)x}$,

(1.5) $V_{U(x)y,z} = V_{x,y}V_{x,z} - U_x U_{y,z}, \quad V_{z,U(x)y} = V_{z,x}V_{y,x} - U_{z,y}U_x$,

(1.6) $U_{U(x)y,z} = V_{x,y}U_{x,z} - U_x V_{y,z} = U_{x,z}V_{y,z} - V_{z,y}U_x$,

(1.7) $V_{x,y}U_z + U_z V_{y,x} = U_{\{xyz\},z}, \quad V_x U_z + U_z V_x = U_{x \circ z,z}$,

(1.8) $U_x U_y U_z + U_z U_y U_x + V_{x,y}U_z V_{y,x} = U_{\{xyz\}} + U_{U(x)U(y)z,z}$.

We always have an expression for the J-ideal generated by K.

1.9 LEMMA. *The ideal in J generated by a subspace K is the outer hull $\overline{K} = \sum_{n=0}^{\infty} U_{\hat{J}}^n L$ of $L = K + U_K \hat{J}$. If $K \lhd I \lhd J$, then $L \lhd I$ with $U_L \hat{I} \subset K$.*

PROOF. Clearly K and L generate the same ideal in J since this must contain $U_K \hat{J}$ by J-innerness. Clearly $\sum U_{\hat{J}}^n L$ is an outer ideal in J containing L, and it is an inner ideal, since for the spanning elements $t = U_{a_1} \cdots U_{a_n} s$ ($s = k$ or $U_k a$, $k \in K$, $a_i \in \hat{J}$) we have $U_t \hat{J} = U_{a_1} \cdots U_{a_n} U_s U_{a_n} \cdots U_{a_1} \hat{J}$ (by (0.3)) $\subset U_{\hat{J}}^n U_K \hat{J}$ (since $U_s = U_k$ or $U_k U_a U_k$ by (0.3)). Thus $\overline{K}$ is as given.

L is I-outer since by (1.7), (1.8) $V_I(U_K \hat{J}) = \{U_{I \circ K,K} - U_K V_I\}\hat{J} \subset U_K \hat{J}$ (by $K \lhd I$) and $U_I(U_K \hat{J}) = \{U_{I \circ K} + U_{U(I)K,K} - U_K U_I - (U_{I \circ K,K} - U_K V_I)V_I\}\hat{J} \subset U_K \hat{J}$ (by $K \lhd I$). To see L is I-inner it will suffice to show $U_L \hat{I} \subset K$. $U_L \hat{I}$ is spanned by all $U_t \hat{I}$ and $U_{t,s}\hat{I}$ for t, s of the form k or $U_k a$ ($k \in K$, $a \in \hat{J}$). Here $U_k \hat{I}$, $U_{U(k)a}\hat{I} = U_k U_a U_k \hat{I} \subset U_k I$ (by (0.3) and $k \in I \lhd J$), $U_{k,I}\hat{I}$, and $U_{U(k)a,I}\hat{I} = \{-U_{U(k,I)a,k} + U_k V_{a,I} + U_{k,I}V_{a,k}\}\hat{I}$ (by linearized (0.2)) $\subset -U_{I,k}\hat{I} + U_k I + U_{k,I} I$ all fall in $K \lhd I$. ■

In order to show $U_I U_I \overline{K} \subset K$, we need to know how $U_I + V_{I,I}$ interacts with U_J.

1.10 LEMMA. *If $I = I^3 \lhd J$ and $\mathfrak{M}$ denotes the unital subalgebra of* End(J) *generated by U_I, $V_{I,I}$, V_I, then*

(1.11) $V_{I,\hat{J}} + V_{\hat{J},I} \subset V_{I,I}$.

(1.12) $U_I V_J \subset \mathfrak{M} V_{I,I}$,

(1.13) $\{U_I + V_{I,I}\} U_{\hat{J}} \subset \mathfrak{M}\{U_I + V_{I,I}\}\{V_{\hat{J}} + 1\}$,

(1.14) $2\{U_I + V_{I,I}\} V_{\hat{J}}^2 \subset \mathfrak{M}\{U_I + V_{I,I}\}\{D_{\hat{J},\hat{J}} + V_{\hat{J}} + 1\}$.

PROOF. For convenience we may assume $J = \hat{J}$ is unital and avoid a plethora of hats. Then for (1.11) we have $V_I = V_{I,I} \subset V_{I,J}$ by (1.1), and $V_{I,J} = V_{U(I)I,J} \subset -V_{U(I)J,I} + V_{I,\{IIJ\}}$ (by linearized (1.4)) $\subset V_{I,I}$ (since $I \lhd J$), and, dually, $V_{J,I} \subset V_{I,I}$.

For (1.12) we have for $x \in I$, $a \in J$ that $U_x V_a = U_x U_{a,1}$ (by (1.1)) $= V_{x,a} V_{x,1} - V_{U(x)a,1}$ (by (1.5)) $\in V_{I,J} V_{I,J} - V_{I,J}$ (since $I \lhd J$) $\subset V_{I,I}^2 + V_{I,I}$ (by (1.11)) $\subset \mathfrak{M} V_{I,I}$.

For $V_{I,I} U_J$ in (1.13) we have for $x, y \in I$, $a \in J$ that $V_{x,y} U_a = U_{a,x} V_{y,a} - U_{U(a)y,x}$ (by (1.6)) $\subset U_{J,I} V_{I,J} - U_{I,I} \subset U_{J,I} V_{I,I} + U_{I,I}$ (by (1.11)) $\subset -V_{I,I} U_{J,I} + U_{\{IIJ\},I} + U_{\{III\},J} + U_{I,I}$ (by linearized (1.7)) $\subset \mathfrak{M} U_{I,J} \subset \mathfrak{M}\{V_I V_J - V_{I,J}\}$ (by (1.3)) $\subset \mathfrak{M}\{V_{I,I} V_J + V_{I,I}\}$ (by (1.11)). For $U_I U_J$ in (1.13) we have U_I spanned by all $U_{U(x)y} = U_x U_y U_x$ and $U_{U(x)y,U(x')y'} = U_{U(x)y,z}$ ($x, y, x', y', z \in I$) since $I = U(I)I$, where $U_x U_y U_x U_a = U_x\{-U_a U_x U_y - V_{a,x} U_y V_{x,a} + U_{\{axy\}} + U_{U(a)U(x)y,y}\}$ (by (1.8)) $= -U_{U(x)a} U_y - U_{U(x)a,x} U_y V_{x,a} + U_x U_{\{axy\}} + U_x U_{U(a)U(x)y,y}$ (by (0.3), (0.2)) $\subset \mathfrak{M} U_I + \mathfrak{M} V_{I,J}$ (since $U_x a$, $\{axy\}$, $U_a U_x y \in I \lhd J$) $\subset \mathfrak{M}\{U_I + V_{I,I}\}$ (by (1.11)) and $U_{I,I} U_J \subset V_{I,J} V_{I,J} - V_{I,U(J)I}$ (by (1.5)) $\subset V_{I,I}^2 - V_{I,I}$ (by (1.11)) $\subset \mathfrak{M} V_{I,I}$.

For (1.14) we have by (0.5), (1.3) that $2V_a V_b = [V_a, V_b] + (V_a V_b + V_b V_a) = D_{a,b} + (2U_{a,b} + V_{a \circ b}) \in D_{J,J} + 2U_J + V_J$, so $2\{U_I + V_{I,I}\} V_J^2 \subset \{U_I + V_{I,I}\}\{D_{J,J} + V_J + U_J\} \subset \mathfrak{M}\{U_I + V_{I,I}\}\{D_{J,J} + V_J + 1\}$ (by (1.13)). ■

1.15 REMARK. In some cases a subideal is necessarily an ideal: an idempotent subideal is an ideal. This holds, in particular, if the subideal K has a covering family of idempotent $\{e_i\}$ ($K = \Sigma U_{e_i} K$), thus if K has a unit (in this case K is a direct summand).

PROOF. It suffices to work our way up the subideal chain $K \lhd I_1 \lhd I_2 \lhd \cdots \lhd I_n \lhd J$ one step at a time, i.e. to prove $K \lhd I \lhd J \Rightarrow K \lhd J$. For convenience we may assume that J itself is unital. If K is idempotent, then $K = U_K K$ is J-outer since $U_J(U_K K) = \{-U_K U_J - V_J U_K V_J + U_{K \circ J} + U_{U(J)K,K}\}K$ (by (1.8)) $\subset U_K I - \{U_{J \circ K,K} - U_K V_J\} I + U_I K + U_{I,K} K$ (by (1.7)) $\subset U_K I + U_I K + U_{I,K} I \subset K$, and it is J-inner since, for the spanning elements $t = U_k k'$ of $K = U_K K$, we have $U_t J = U_k U_{k'} U_k J \subset U_k I \subset K$.

Note that if K has a covering family of idempotents $\{e_i\}$ it is idempotent since $K = \Sigma U_{e_i} K \subset U_K K$. One can also show $K \lhd J$ directly from properties of Peirce decompositions. If $K = U_e K$ is unital, we actually have K a direct summand of J, $J = K \boxplus K'$ for $K = U_e J$, $K' = U_{1-e} J$. ■

2. Minimal ideals. We remarked that a minimal ideal $I \lhd J$ is either trivial or idempotent. We want to show that the idempotent minimal ideals I are at least $\mathfrak{D}$-simple, and we do this by showing that proper $\mathfrak{D}$-invariant I-ideals would lead to J-ideals properly contained in I.

2.1 THEOREM. *If $I = I^3$ is an idempotent ideal in J and $K \lhd I$ is a proper ideal such that either* (i) *$2I = I$ and K is invariant under $D_{J,J} = [V_J, V_J]$, or* (ii) *$2I = 0$ and K is invariant under $D_J = V_J$, then K generates a proper $D_{J,J}$ or D_J-invariant ideal $\bar{K} \lhd J$.*

PROOF. As usual, we may assume J is unital. By (1.9) we have $\bar{K} = \sum U_J^n L$ for $L = K + U_K J \lhd I$ with $U_L \hat{I} \subset K$. We will show that in both cases $U_I U_I \bar{K} \subset L$. This shows $\bar{K}$ is proper if K is: if $\bar{K} = I$, then by idempotence $I = U_I U_I I = U_I U_I \bar{K} \subset L$ forces $I = L$, hence $I = U_I I = U_L I \subset K$ forces $K = I$. By (0.4) $\bar{K}$ is invariant under any derivation that K is.

In the notation of (1.10) we have

$$(*) \qquad U_I \bar{K} = \sum U_I U_J^n L \subset \sum \mathfrak{M}(U_I + V_{I,I}) V_J^m L$$

using (1.13) to convert U_J's into V_J's, and then using (1.7) to move the resulting V_J's to the right. In case (ii) K (and hence L too, using (1.7)) is invariant under V_J, so $(*)$ yields $U_I \bar{K} \subset \sum \mathfrak{M}(U_I + V_{I,I}) L \subset L$ since $L \lhd I$, and $U_I \bar{K} \subset L$ in this case. In case (i) $(*)$ yields only $U_I \bar{K} \subset \mathfrak{M}(U_I + V_{I,I})(L + V_J L)$: here $I = 2I$ implies $U_I + V_{I,I} \subset 2(U_I + V_{I,I})$, so we can use (1.14) repeatedly to convert higher powers $V_J^m L$ ($m \geqslant 2$) into lower powers plus $D_{J,J}$'s, then using $[D, V_z] = V_{D(z)}$ from (0.4) to move the resulting D's to the right, and then absorbing these into L since $D(L) = D(K) + D(U_K J) \subset D(K) + U_{D(K),K} J + U_K D(J)$ (by (0.4)) $\subset K + U_K J = L$ (by the assumed invariance of K under $D_{J,J}$). But then $U_I \bar{K} \subset \mathfrak{M}(L + V_J L) \subset L + V_J L$ (because $L \lhd I$ and $\mathfrak{M} V_J \subset (V_J + 1)\mathfrak{M}$ since $U_I V_J \subset \mathfrak{M}$ by (1.12), $V_I V_J \subset V_{I,I} V_J$ by (1.11), and $[V_{I,I}, V_J] = V_{\{IIJ\}} - V_{J, I \circ I} \subset \mathfrak{M}$ by (1.7), (1.2), (1.11), so $U_I U_I \bar{K} \subset U_I(L + V_J L) \subset \mathfrak{M} L$ (by (1.12)) $\subset L$ (by $I \lhd L$). Thus in both cases we have $U_I U_I \bar{K} \subset L$, as desired. ■

In characteristic 2 the maps $D_x = V_x$ are derivations too.

2.2 LEMMA. *If $2I = 0$ then V_a induces a derivation of I for each $a \in J \rhd I$.*

PROOF. V_a leaves $I \lhd J$ invariant, and on I it satisfies conditions (0.4) since $D(x^2) = x^2 \circ a = x \circ (x \circ a) - 2U_x a$ (by (1.3)) $= x \circ (x \circ a)$ (since $2U_x a \in 2I = 0$) $= x \circ D(x)$ and $D(U_x y) = V_a U_x y = \{-U_x V_a + U_{a \circ x, x}\} y$ (by (1.7)) $= U_x D(y) + U_{D(x),x} y$ (since $-a \circ y = a \circ y$ for $y \in I$ because $2I = 0$). ■

Now we can prove our main result about subideals of idempotent ideals.

2.3 THEOREM. *A minimal ideal $I \lhd J$ in a quadratic Jordan algebra is either trivial ($I^3 = 0$) or $\mathfrak{D}$-simple (indeed, has no proper ideals invariant under those inner derivations induced on I by J).*

PROOF. In order to apply 2.1 we must show $2I = I$ or 0; but $2I$ is again an ideal of J, so by minimality of I it can only be I or 0. ■

The *heart* of a Jordan algebra is the intersection $\mathcal{H}(J) = \bigcap I$ of all nonzero ideals I. If it is nonzero it is the unique minimal ideal, contained in all other ideals, so

2.4 THEOREM. *The heart $\mathcal{H}(J)$ of a quadratic Jordan algebra J is either trivial or $\mathfrak{D}$-simple.* ■

In Zelmanov's work [5] the heart played a major role: the heart of an i-exceptional prime algebra was actually simple with capacity, from which it follows that the prime i-exceptional algebra was itself essentially a simple Albert algebra.

The result we want, of course, is that a minimal ideal is trivial or (ordinary) simple. This holds for associative and alternative algebras by work of Slater and

Zhevlakov [**4**]. It can be deduced for certain Jordan algebras from Block's characterization [**1**] of differentiably simple algebras.

In [**3**] it was shown that the middle nucleus and center coincide for $\mathfrak{D}$-semiprime linear Jordan algebras; since $\mathfrak{D}$-simple algebras are $\mathfrak{D}$-semiprime, we get

2.5 PROPOSITION. *If I is a nontrivial minimal ideal in a linear Jordan algebra, then* $\mathfrak{N}_m(I) = \mathfrak{C}(I)$. ■

REFERENCES

1. R. Block, *Determination of the differentiably simple rings with a minimal ideal*, Ann. of Math. (2) **90** (1969), 433–459.

2. N. Jacobson, *Structure theory of Jordan algebras*, Univ. of Arkansas Lecture Notes, Fayetteville, 1981.

3. K. McCrimmon and Ng Seong Nam, *Middle nucleus = center in semiprime Jordan algebras*, Proc. Amer. Math. Soc. **86** (1982), 21–24.

4. M. Slater, *The socle of an alternative ring*, J. Algebra **14** (1970), 443–463.

5. E. I. Zelmanov, *On prime Jordan rings*, Algebra i Logika **18** (1979), 162–175.

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