SIMULTANEOUS ATTAINABILITY OF CENTRAL LYAPUNOV AND BOHL EXPONENTS FOR ODE LINEAR SYSTEMS

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Abstract. Millions'ev's Accessibility Theorem for the central Lyapunov exponent of a linear ODE system is extended to simultaneous attainability of both central Lyapunov and Bohl exponents.

1. Let

\( \dot{x} = A(t)x, \quad t \geq 0, x \in \mathbb{R}^n, \|A(t)\| \leq a_0. \)

The Lyapunov exponent \( \lambda(x) \) and Bohl exponent \( \beta(x) \) of a solution \( x(t) \) are given by

\[
\lambda(x) = \lim_{t \to -\infty} \frac{1}{t} \ln |x(t)|, \quad \text{resp.} \quad \beta(x) = \lim_{t \to -\infty} \frac{1}{t-s} \ln \left| \frac{x(t)}{x(s)} \right|.
\]

(In fact these are upper exponents; the lower ones are defined similarly, with \( \lim \) in place of \( \lim \).

In general neither these exponents nor their suprema \( \lambda_0 = \sup_x \lambda(x), \beta_0 = \sup_x \beta(x) \) are stable under small perturbations of the system. Instead the so-called central Lyapunov exponent\(^1 \) \( \Lambda \geq \lambda_0 \) and Bohl exponent \( B \geq \beta_0 \) can be defined being stable upward (resp. lower exponents being stable downward). To introduce them and to describe exactly this “upward stability” we need a notion of upper functions (for brevity we omit similar notions and results about lower exponents).

2. Let \( X(t, s) = X(t)X^{-1}(s) \) where \( X(t) \) is a fundamental matrix of (1). As is known,

\[
|X(t, s)| \leq e^{a|t-s|}
\]

and

\[
|X(t, s)| = \max_x \left| \frac{x(t)}{x(s)} \right|
\]

where \( \max \) is taken over all nonzero solutions of (1).

**Definition.** A bounded function \( K(t) \) is an upper function for system (1) if there is a constant \( D = D_K \) such that

\[
|X(t, s)| \leq D e^{\int_t^s K(\alpha) d\alpha} \quad (t \geq s).
\]

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\(^1\) More popular notation is \( \Omega \) rather than \( \Lambda \).
For example, by (2), \( K(t) = a_0 \) is an upper function with \( D = 1 \). Let
\[
\bar{K} = \lim_{t \to -\infty} \frac{1}{t} \int_0^t K(\alpha) d\alpha, \quad \overline{K} = \lim_{t \to -\infty} \frac{1}{t} \int_t^s K(\alpha) d\alpha.
\]

**Definition.** The central Lyapunov exponent \( \Lambda \), resp. Bohl exponent \( \beta \) is given by
\[
\Lambda = \inf \bar{K}, \quad \beta = \inf \overline{K},
\]
where the inf is taken over all upper functions.

It is easily seen that \( \lambda_0 \leq \Lambda, \beta_0 \leq \beta \) and \( \Lambda \leq \beta \).

3. Consider a perturbed system
\[
y = \left[ A(t) + \tilde{A}(t) \right] y
\]
and let its upper functions and exponents be marked by \( \tilde{\cdot} \).

The upward stability of \( K(t), \Lambda, \beta \) means that given \( \varepsilon > 0 \) there is \( \delta = \delta(\varepsilon) > 0 \) such that if \( |\tilde{A}(t)| \leq \delta \), then
\[
\tilde{K}(t) \leq K(t) + \varepsilon, \quad \tilde{\Lambda} \leq \Lambda + \varepsilon, \quad \tilde{\beta} \leq \beta + \varepsilon.
\]

The next theorem is well known [1].

4. **Theorem.** \( K(t), \Lambda, \) and \( \beta \) are always upward stable.

**Proof.** It suffices to prove \( \tilde{K}(t) \leq K(t) + \varepsilon \); then the rest follows by (5), (6). Let
\[
Y(t, s) = Y(t)Y^{-1}(s) \quad \text{where} \quad Y(t) \text{ is a fundamental matrix of (7).}
\]
By the Variation of Constants Formula,
\[
Y(t, s) = X(t, s) + \int_s^t X(t, \tau) \tilde{A}(\tau) Y(\tau, s) d\tau.
\]

Take norms, use (4) and set
\[
|Y(t, s)| = De^{\int_s^t K(\alpha) d\alpha} u(t).
\]
Then
\[
u(t) \leq 1 + \int_s^t D|\tilde{A}(\tau)| u(\tau) d\tau
\]
and by Gronwall's inequality, \( u(t) \leq \exp \int_s^t D|A(\tau)| d\tau \). Now, if \( |\tilde{A}(t)| \leq \delta \), then by (8) \( \tilde{K}(t) = K(t) + D\delta \) is upper for (7). So \( \delta(\varepsilon) = \varepsilon / D \).

In particular Theorem 4 implies that if \( \lambda_0 = \lambda \) (or \( \beta_0 = \beta \)), then \( \lambda_0 \) (or \( \beta_0 \)) is itself stable up. As is known, for a constant system (1) (i.e. \( A(t) = \text{const} \)) one has always \( \lambda_0 = \beta_0 = \Lambda = \beta \), and so all exponents are stable up.

5. In contrast, for nonautonomous systems the central exponents \( \Lambda \) and \( \beta \) need not be attainable by individual solution exponents, i.e. it may happen that \( \lambda_0 \leq \Lambda \) and/or \( \lambda_0 < \beta \) (as well as \( \Lambda < \beta \)). However the Accessibility Theorem [2] states that the central Lyapunov exponent \( \Lambda \) is always attainable by means of arbitrarily small perturbations in the following sense: given \( \delta > 0 \) there is a perturbation with \( |\tilde{A}(t)| < \delta \) such that \( \tilde{\lambda}_0 \geq \Lambda \) for the perturbed system (7).

It turns out that this theorem can be extended to the attainability of \( \beta \); moreover, a simultaneous attainability of both \( \Lambda \) and \( \beta \) can be established and at the same time the original proof [2] can be considerably shortened.
6. **Theorem.** Let system (1) have central Lyapunov exponent $\Lambda$ and Bohl exponent $B$. Given $\delta_0 > 0$ there is a perturbation $\tilde{A}(t)$ with $|\tilde{A}(t)| < \delta_0$ such that system (6) has a solution $y(t)$ with both $\lambda(y) \geq \Lambda$ and $\beta(y) \geq B$.

To prove this theorem we start with a technical remark and a number of lemmas.

7. **Remark.** All the above definitions of exponents or upper functions are given with continuously varying $t$ and $s$. But nothing will be changed if we replace them by discrete variables $t_n = nT$, $s_m = mT$, where $T > 0$ is fixed and $m, n = 1, 2, \ldots$. This follows by the fact that by (2), $|X(t, s)| \leq e^{\alpha_0 T} = \text{const}$ as well as $|x(t)|/|x(s)| \leq e^{\alpha_0 T} = \text{const}$ for $|t - s| \leq T$, so that any difference between continuous $t$ and discrete $t_n \leq t < t_{n+1}$ vanishes by taking $\lim$ or else is absorbed by the constant $D$ in (4). In particular, $K(t)$ remains upper if (4) holds just for $t = t_n$, $s = s_m$.

8. **Lemma.** Let $T > 0$ be fixed, $t_n = nT$, $J_n = [t_{n-1}, t_n]$, $n = 0, 1, \ldots$ and

$$\ln |X(t, s)| = f(t, s), \quad \text{i.e.,} \quad |X(t, s)| = e^{f(t, s)}.$$

Define a step function $K(t)$ by

$$K(t) = \lambda_n = \frac{1}{T} f(t_n, t_{n-1}) \quad \text{on } J_n, n = 1, 2, \ldots$$

(the illegal "double definition" at $t = t_n$ can be neglected). Then $K(t)$ is an upper function and hence $K \geq \Lambda$, $K \geq B$.

**Proof.** By (2), $K(t)$ is bounded: $|K(t)| \leq a_0$. Since $X(t, s) = X(t, r)X(r, s)$, we have $f(t, s) \leq f(t, r) + f(r, s)$, and since

$$f(t_k, t_{k-1}) = \lambda_k T = \int_{t_{k-1}}^{t_k} K(\alpha) \, d\alpha,$$

we have for $t = t_n$, $s = s_m$, $n \geq m$,

$$f(t, s) \leq \sum f(t_k, t_{k-1}) = \int_s^T K(\alpha) \, d\alpha, \quad \text{i.e.} \quad |X(t, s)| \leq e^{\int_s^T K(\alpha) \, d\alpha}.$$

By Remark 7, $K(t)$ is upper.

The next several lemmas constitute so-called Millionshchikov’s Rotation Method. It can be found in [2], that is why we mostly restrict ourselves to some brief outlines of the proofs. Recall that the angle $\gamma = \angle (a, b)$ between two vectors $a, b \in \mathbb{R}^n$ is given by $\cos \gamma = a \cdot b/(|a| \cdot |b|)$, $0 \leq \gamma \leq \pi$.

9. **Lemma.** Let $a$ and $c$ be vectors in $\mathbb{R}^n$ with $|a| = |c|$ and $\angle (a, c) = \gamma \neq 0, \pi$. Then there is a unitary operator $U(t)$: $\mathbb{R}^n \rightarrow \mathbb{R}^n$ defined on a given interval $J^*$: $t^* \leq t \leq t^* + T$, $T \geq 1$, such that

(i) $U(t^*) = I$, $U(t^* + T)a = c$,

(ii) $|U(t) - I| = |U^{-1}(t) - I| \leq \gamma,

(iii) $|U(t)U^{-1}(t)| \leq \gamma$.

**Sketch of Proof.** Let $V(\omega)$: $\mathbb{R}^n \rightarrow \mathbb{R}^n$ be the rotation by the angle $\omega$ from $a$ to $c$ in the 2-plane $P_{ac}$ spanned by $a, c$, and $V(\omega) = \text{identity}$ on the orthogonal complement to $P_{ac}$. Then $V(\omega)$ is unitary and in a proper orthonormal basis of $\mathbb{R}^n$ (the two
first elements in $P_{\omega}$ the matrix of $V(\omega)$ is 

$$\text{diag}\left\{ \begin{pmatrix} \cos \omega & \sin \omega \\ -\sin \omega & \cos \omega \end{pmatrix}, 1, \ldots, 1 \right\}.$$

Set $U(t) = V[(t - t^*)\gamma/T]$. Then (i) is clear and (ii), (iii) follow by direct computation.

10. **Lemma.** Let $x(t)$ be a solution of (1) considered on an interval $J^* = [t^*, t^* + T]$. $T \geq 1$. Next, let $x(t^* + T) = a$, and $c$ be a vector with $|c| = |a|$ and $\langle a, c \rangle = \gamma \neq 0, \pi$. Then there is a perturbation $\tilde{A}(t)$ with norm

$$|\tilde{A}(t)| \leq \gamma(2a + 1)$$

such that the perturbed system (7) has a solution $y(t)$ with

$$y(t^*) = x(t^*) \quad \text{and} \quad y(t^* + T) = c \quad \text{(so that} \quad |y(t^* + T)| = |x(t^* + T)|).$$

**Proof.** Let $y(t) = U(t)x(t)$ where $U(t)$ is as in Lemma 9. Then clearly (11) holds. Next, $y = U\dot{x} + \dot{U}x = (UAU^{-1} + UUU^{-1})y = (A + \tilde{A})y$ where $\tilde{A} = UAU^{-1} - A + UUU^{-1}$. By Lemma 9, $|UU^{-1}| \leq \gamma$ and

$$|UAU^{-1} - A| \leq |UA(U^{-1} - I)| + |(U - I)A| \
\leq |UA| \cdot \gamma + \gamma |A| \
= 2\gamma |A| \quad \text{(since $U$ is unitary, $|UA| = |A|$)}.$$

Now (10) follows.

11. **Lemma.** Let $a, b, c$ be three coplanar vectors in $\mathbb{R}^n$ such that $|a| = |b| = |c|$ and $0 \leq \gamma \leq \theta$ where $\gamma = \angle (a, c), \theta = \angle (a, b)$. Then $c = \alpha a + \beta b$ where

$$\beta = \frac{\sin \gamma}{\sin \theta} > 0 \quad \text{and} \quad \alpha = \frac{\sin(\theta - \gamma)}{\sin \theta} > 0.$$

Proof is by direct computation.

12. **Proof of Theorem 6.** Choose first $\gamma$ and $T$ as follows. Let $\delta = \delta_0/2$. Fix $\gamma$ with

$$0 < \gamma \leq \delta/(2a + 1).$$

Then fix $T \geq 1$ and so large that $\sin \gamma \geq 2e^{-\delta T}$, i.e.

$$\sin \gamma - e^{-\delta T} \geq e^{-\delta T}.$$

Define $K(t)$ as in Lemma 8 and classify the solutions $x(t)$ of system (1) on each interval $J_n = [t_{n-1}, t_n]$ as follows.

If

$$\begin{cases} 
|x(t)| = e^{\lambda_n T}, & \text{then $x(t)$ is maximal on $J_n$}, \\
|x(t)| \geq e^{(\lambda_n - \delta)T}, & \text{then $x(t)$ is rapid on $J_n$}, \\
|x(t)| < e^{(\lambda_n - \delta)T}, & \text{then $x(t)$ is slow on $J_n$}.
\end{cases}$$

Notice that a maximal solution always exists by (3) and (9). Since a constant multiple of $x(t)$ falls into the same category as $x(t)$, we can normalize $x(t)$ as we like without change of its category.
Now we are going to perturb system (1) inductively on each interval $J_1, J_2, \ldots$.
Each time the perturbation $\tilde{A}(t)$ will be found by Lemma 10 and hence with $|\tilde{A}(t)| \leq \delta$ by virtue of (10) and (13). We will not mention this smallness any longer.
Starting with a rapid solution on $J_n$ we will watch its behavior on $J_{n+1}$ and depending on that choose a perturbation on $J_n$ (but not on $J_{n+1}$ yet).

1st step. Pick a maximal solution $x(t)$ on $J_1$. Then it is also rapid

$$\frac{|x(t_1)|}{|x(t_0)|} = e^{\lambda_1 T} > e^{(\lambda_1 - \delta) T} \quad (t_0 = 0).$$

Look at its natural extension to $J_2$. If it remains rapid on $J_2$, i.e.

$$\frac{|x(t_2)|}{|x(t_1)|} \geq e^{(\lambda_1 - \delta) T},$$

then put $\tilde{A}(t) \equiv 0$ on $J_1$, relabel $x(t)$ by $y(t)$ on $J_1$, and the 1st step is completed. As a result we have

$$|x(t_1)| \leq |y(t_1)| \leq |y(t_2)| \leq |x(t_2)| \leq e^{(\lambda_1 - \delta) T} |x(t_0)|,$$

where $x(t)$ is a natural (unperturbed) extension of $y(t)$.

Suppose $x(t)$ is slow on $J_2$ and let $x(t_1) = a$. Find a maximal solution $\xi(t)$ on $J_2$ and normalize it so that the vector $\xi(t_1) = b$ has norm $|b| = |a|$. Since $x(t)$ is slow while $\xi(t)$ is maximal, they cannot be proportional; therefore $\langle a, b \rangle \neq 0, \pi$.
Define a vector $c$ like this: $c = b$ if $\langle a, b \rangle < \gamma$, otherwise let $c = aa + \beta b$ be as in Lemma 11.

Now perturb system (1) on $J_1$ as in Lemma 10. This yields a solution $y(t)$ of (7) with $y(t_0) = x(t_0), |y(t_1)| = |x(t_1)|$ and hence with

$$\frac{|y(t_1)|}{|y(t_0)|} \geq e^{(\lambda_1 - \delta) T}, \quad \frac{|x(t_2)|}{|x(t_1)|} \geq e^{(\lambda_2 - \delta) T}$$

where $x(t)$ is a natural (unperturbed) extension of $y(t)$.

Now perturb system (1) on $J_1$ as in Lemma 10. This yields a solution $y(t)$ of (7) with $y(t_0) = x(t_0), |y(t_1)| = |x(t_1)|$ and hence with

$$\frac{|y(t_1)|}{|y(t_0)|} \geq e^{(\lambda_1 - \delta) T}.$$

The crucial point is that the natural (unperturbed) extension $z(t)$ of $y(t)$ beyond $t_1$ is rapid on $J_2$. Indeed, if $c = b$, then $z(t) = \xi(t)$ which is even maximal on $J_2$.
Otherwise $z(t_1) = c = \alpha a + \beta b = ax(t_1) + \beta \xi(t_1)$ and by linearity

$$z(t) = ax(t) + \beta \xi(t), \quad t \geq t_1.$$

At $t = t_1$ all three norms of $z, x, \xi$ are equal, therefore

$$\frac{|z(t_2)|}{|z(t_1)|} = \frac{|ax(t_2) + \beta \xi(t_2)|}{|\xi(t_1)|} \geq \frac{|\xi(t_2)|}{|\xi(t_1)|} \left( \beta - \alpha \frac{|x(t_2)|}{|\xi(t_2)|} \right).$$

Since $\xi$ is maximal and $x$ is slow on $J_2$, we have

$$\frac{|x(t_2)|}{|\xi(t_2)|} = \frac{|x(t_2)|}{|\xi(t_2)|} \left( \frac{|x(t_1)|}{|\xi(t_1)|} \right) < e^{(\lambda_2 - \delta) T} = e^{-\delta T}. $$
Then
\[ \beta - \alpha \frac{|x(t_2)|}{|\xi(t_2)|} > \frac{\sin \gamma - \sin(\theta - \gamma)e^{-\delta T}}{\sin \theta} \quad (\text{by (12)}) \]
\[ \geq \sin \gamma - e^{-\delta T} \]
\[ \geq e^{-\delta T} \quad (\text{by (14)}). \]

Combining with (15), \( z \) is rapid on \( J_2 \)
\[ \frac{|z(t_2)|}{|z(t_1)|} \geq e^{(\lambda_2 - \delta)T}. \]

Relabeling \( z(t) \) by \( x(t) \) on \( J_2 \), we come again to (15), and the 1st step is entirely completed.

Suppose we have already completed \( m - 1 \) steps of the induction with the following results:

(i) The system is properly perturbed on \( J_1 \cup \cdots \cup J_{m-1} \) but unperturbed yet on \( J_m = [t_{m-1}, t_m] \) or further.

(ii) There is a solution \( y(t) \) of the perturbed system on \( J_1 \cup \cdots \cup J_{m-1} \) with natural (unperturbed) continuous extension \( x(t) \) on \( J_m \) such that
\[ \begin{align*}
|y(t_k)| &\geq e^{(\lambda_k - \delta)T}, \quad k = 1, \ldots, m-1, \\
|y(t_{k-1})| &\geq e^{(\lambda_k - \delta)T}.
\end{align*} \]

\( m \)th step is now exactly as the 1st one, just with \( t_{m-1}, t_m, t_{m+1} \) in place of \( t_0, t_1, t_2 \). Namely, if \( x(t) \) remains rapid on \( J_{m+1} \), then we set \( \bar{A}(t) \equiv 0 \) on \( J_m \), relabel \( x(t) \) by \( y(t) \) on \( J_m \) and so get (17a,b) with \( m \) replaced by \( m + 1 \). In this case the \( m \)th step is completed.

If \( x(t) \) is slow on \( J_{m+1} \), then let \( x(t_m) = a \), find a maximal solution \( \xi(t) \) on \( J_{m+1} \) with \( \xi(t_m) = b, \ |b| = |a| \), and define \( c \) as before: \( c = b \) if \( \xi(a, b) \leq \gamma \), otherwise \( c = aa + \beta b \) as in Lemma 11. Now perturb the system on \( J_m \) as in Lemma 10. This creates a solution \( y(t) \) with \( y(t_{m-1}) = x(t_{m-1}), \ |y(t_m)| = |x(t_m)| \) and hence, by (17b), with
\[ \frac{|y(t_m)|}{|y(t_{m-1})|} \geq e^{(\lambda_m - \delta)T}. \]

As before, the unperturbed continuous extension \( z(t) \) of \( y(t) \) beyond \( t_m \) is rapid on \( J_{m+1} \), and relabeling \( z(t) \) by \( x(t) \) gives again (17a,b) with \( m \) replace by \( m + 1 \). The \( m \)th step is entirely completed.

By induction, we obtain a system (7) defined for all \( t \geq 0 \) with perturbation \( \bar{A}(t) \) of smallness \( |\bar{A}(t)| \leq \delta = \delta_0/2 \) and having a solution \( y(t) \) which satisfies (17a) for all \( k = 1, 2, \ldots \). By the very definition (9) of \( K(t) \),
\[ \int_{t_{n-1}}^{t_n} K(\alpha) \, d\alpha = \lambda_n T, \quad \int_{t_{n-1}}^{t_n} [K(\alpha) - \delta] \, d\alpha = (\lambda_n - \delta) T, \]
so that (17a) implies for $s = t_m, t = t_n (t \geq s)$

$$\frac{|y(t)|}{|y(0)|} \geq \exp \int_0^t [K(\alpha) - \delta] \, d\alpha \quad \text{and} \quad \frac{|y(t)|}{|y(s)|} \geq \exp \int_s^t [K(\alpha) - \delta] \, d\alpha.$$ 

It follows by Remark 7 and Lemma 8 that the Lyapunov and Bohl exponents of $y(t)$ satisfy $\lambda(y) \geq \bar{K} - \delta \geq \Lambda - \delta$ and $\beta(y) \geq \bar{K} - \delta \geq B - \delta$ respectively. To complete the proof let $y^*(t) = y(t)e^{\delta t}$. Then $\lambda(y^*) \geq \Lambda$, $\beta(y^*) \geq B$ and $y^*(t)$ satisfies the system with perturbation $\bar{A}(t) + \delta I$ of smallness $2\delta = \delta_0$.

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