THE SINGULAR INTEGRAL CHARACTERIZATION
OF $H^p$ ON SIMPLE MARTINGALES

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Abstract. The singular integral characterization of $H^1$ on simple martingales was
given by S. Janson. We show that his result cannot be extended to $H^p$ if $p (> 0)$ is
very small.

Let $\Omega = (0, 1]$. Let $F$ be the $\sigma$-field of all Borel sets in $\Omega$. Let $dx$ be the Lebesgue
measure. Then $(\Omega, F, dx)$ is a probability space. Let $d \geq 3$ be an integer. For each
integer $n \geq 0$, let $F_n$ be the sub-$\sigma$-field of $F$ generated by $((k - 1)d^{-n}, kd^{-n}]$, $k = 1, \ldots, d^n$. Set

$I(k_1, \ldots, k_n) = ((k_1 - 1)d^{-1} + \cdots + (k_{n-1} - 1)d^{-1-n} + (k_n - 1)d^{-n},$

$\quad (k_1 - 1)d^{-1} + \cdots + (k_{n-1} - 1)d^{-1-n} + k_n d^{-n}]$

for each $k_1, \ldots, k_n \in \{1, \ldots, d\}$.

A martingale is a sequence of complex-valued integrable functions $\{f_n\}_{n=0}^\infty$ such
that $E[f_{n+1} \mid F_n] = f_n$, where $E[\cdot \mid F_n]$ denotes the conditional expectation with
respect to the sub-$\sigma$-field $F_n$. We write $f$ for $\{f_n\}_{n=0}^\infty$. If $f$ is generated from a function
$\tilde{f}(x) \in L^1(\Omega)$ by

$$f_n = E[\tilde{f} \mid F_n],$$

we identify $\tilde{f}$ and $f$.

For a martingale $f$ we define

$$f^*(x) = \sup_{n \geq 0} |f_n(x)|.$$

For $p \in (0, \infty)$, $f$ is said to belong to $H^p$ if $\|f^*\|_p < +\infty$, where

$$\|f^*\|_p = \left\{ \int_\Omega (f^*)^p \, dx \right\}^{1/p}.$$

It is well known that if $p > 1$, then $H^p$ and $L^p(\Omega)$ can be identified. That is,$f^* \in L^p(\Omega)$ if and only if there exists a function $\tilde{f}(x) \in L^p(\Omega)$ such that (1) and

$$c_p \|f^*\|_p \leq \|\tilde{f}\|_p \leq \|f^*\|_p,$$

hold. It is also known that if $f^* \in L^1(\Omega)$, then $f$ is generated from an $L^1$-function but
that the converse is not true.

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617

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For \( n \geq 1 \), set \( \Delta f_{n} = f_{n} - f_{n-1} \). Since \( \Delta f_{n} \) is \( F_{n} \)-measurable, the notation \( \Delta f_{n}(I(k_{1}, \ldots, k_{n})) \) makes sense. Let

\[
V = \left\{ \xi = (x_{k})_{k=1}^{d} \in \mathbb{C}^{d} : \sum_{k=1}^{d} x_{k} = 0 \right\},
\]

where \((x_{k})_{k=1}^{d}\) denotes a \( d \)-dimensional column vector and \( \mathbb{C} \) is the set of all complex numbers. Note that \( \left( \Delta f_{n}(I(k_{1}, \ldots, k_{n-1}, k)) \right)_{k=1}^{d} \in V \). Let \( A \) be a linear operator from \( V \) to \( V \). Set

\[
(A_{n} g_{n}(I(k_{1}, \ldots, k_{n-1}, k)))_{k=1}^{d} = A\left( \Delta f_{n}(I(k_{1}, \ldots, k_{n-1}, k)) \right)_{k=1}^{d},
\]

\[
g_{n} = \sum_{k=1}^{n} \Delta g_{k}, \quad g_{0} = 0,
\]

(2)

\[T_{f} = \{ g_{n} \}_{n=0}^{\infty} \text{ and } (T_{f})_{n} = g_{n}.
\]

Assume that \( A_{1}, \ldots, A_{m} \) are one or more linear transformations on \( V \), and let \( T_{1}, \ldots, T_{m} \) be the corresponding operators defined by (2).

The above definitions are due to Janson [9] and Chao and Taibleson [4]. They introduced the above as the analogues of Hardy spaces and singular integral operators on \( \mathbb{R}^{n} \). These have been known to lead to some rewarding feed-backs with the Euclidean case. (See [8, 9 and 12].)

For \( \rho > 0 \) and close to 1, the above definitions work very well. Janson [9] showed

**Theorem A.** There exists \( p_{0}(A_{1}, \ldots, A_{m}) < 1 \) such that

\[
\liminf_{n \to \infty} \left\{ \| f_{n} \|_{p} + \sum_{j=1}^{m} \| (T_{j} f)_{n} \|_{p} \right\} \geq C_{p,A_{1}, \ldots, A_{m}} \| f^{*} \|_{p}
\]

for any \( p > p_{0} \) and any martingale \( f \) if and only if

(4) \( A_{1}, \ldots, A_{m} \) do not have a common real eigenvector, where \( C_{p,A_{1}, \ldots, A_{m}} > 0 \).

(See also Chao and Taibleson [4].)

The hard implication is the "if" part. Janson proved this from the observation that if (4) holds, then \( (\| f_{n} \|^{2} + \sum_{j=1}^{m} |(T_{j} f)_{n}|^{2})^{p_{0}/2} \) becomes a submartingale. (For another argument for the case \( p = 1 \) that does not appeal to the submartingale property, see [11].)

In this note, we show

**Theorem 1.** There exists \( p_{1}(d) > 0 \) such that

\[
\inf \left\{ \limsup_{n \to \infty} \left( \| f_{n} \|_{p} + \sum_{j=1}^{m} \| (T_{j} f)_{n} \|_{p} \right) / \| f^{*} \|_{p} : f \in H^{p}, f \equiv 0, f_{0} = 0 \right\} = 0
\]

for any \( p \in (0, p_{1}(d)] \), any \( m \geq 1 \) and any \( A_{1}, \ldots, A_{m} \), where \( p_{1}(d) \) depends only on \( d \).

As a consequence of Theorem 1, \( p_{0}(A_{1}, \ldots, A_{m}) \) in Theorem A cannot be less than \( p_{1}(d) \), no matter how we choose \( m \) and \( A_{1}, \ldots, A_{m} \). Thus, a "singular integral" characterization of \( H^{p} \) on these simple martingales is impossible for \( p \leq p_{1}(d) \) if we define "singular integrals" by (2). This tells us that \( H^{p} \) theory on these martingales and that on \( \mathbb{R}^{n} \) are not quite parallel if \( p \) is very small.
We get Theorem 1 as a corollary of the following Theorem 2. Let \( H \) be a Hilbert
space. Let

\[
V = \left\{ (x_k)_{k=1}^d : x_k \in H, \sum_{k=1}^d x_k = 0 \right\}.
\]

Let \( A \) be a linear operator from \( V \) to \( V \). Set

\[
(\Delta g_n(I(k_1, \ldots, k_{n-1}, k)))_{k=1}^d = A(\Delta f_n(I(k_1, \ldots, k_{n-1}, k)))_{k=1}^d.
\]

Set

\[
g_0 = 0, \quad g_n = \sum_{k=1}^n \Delta g_k \quad \text{and} \quad g = \{g_n\}_{n=0}^\infty.
\]

Then \( g \) is an \( H \)-valued martingale. Set \( T_f = g \) and \( (T_f)_n = g_n \).

**Theorem 2.** There exists \( p_\d (d) > 0 \) such that

\[
\inf_{n \to \infty} \lim \sup_{n \to \infty} \|/(T_f)_n\|_p/\|f^*_n\|_p : f \in H^p, f \equiv 0, f_0 = 0 = 0.
\]

for any \( p \in (0, p_\d (d)] \) and any \( A \), where \( p_\d (d) \) depends only on \( d \).

For \( \xi = (x_k)_{k=1}^d \in V \), let \( \xi(k) = x_k \). If we substitute \( H = C^{m+1} \) and

\[
A \xi = \begin{pmatrix}
(\xi(1), (A_1 \xi)(1), \ldots, (A_m \xi)(1)) \\
(\xi(2), (A_1 \xi)(2), \ldots, (A_m \xi)(2)) \\
\cdots \\
(\xi(d), (A_1 \xi)(d), \ldots, (A_m \xi)(d))
\end{pmatrix},
\]

then \( A \) is a linear operator from \( V \) to \( V \) and

\[
(Tf)_n = (f_n, (T_1 f)_n, \ldots, (T_m f)_n).
\]

Thus, Theorem 1 follows from Theorem 2.

We now prove Theorem 2.

We define integer-valued functions

\[
\{h_i(k_1, \ldots, k_n)\}_{n=1,2,3, \ldots; i, k_1, \ldots, k_n \in \{1, \ldots, d\}}
\]

inductively by

\[
h_i(k) = \delta(i, k),
\]

\[
h_i(k_1, \ldots, k_{n-1}, k) = h_i(k_1, \ldots, k_{n-1}) - h_{k_{n-1}}(k_1, \ldots, k_{n-1}) \delta(i, k),
\]

where \( \delta(i, k) \) is the Kronecker delta.

Set

\[
\xi^0 = \begin{pmatrix}
1 \\
-1 \\
0 \\
\vdots \\
0
\end{pmatrix} \in V, \quad \Delta f_n(I(k))_{k=1}^d \xi^0 = \Delta f_n(I(k))_{k=1}^d \xi^0,
\]

and

\[
(\Delta f_n(I(k_1, \ldots, k_{n-1}, k)))_{k=1}^d = -h_{k_{n-1}}(k_1, \ldots, k_{n-1}) \xi^0, \quad \text{for } n \geq 2.
\]
From this $\Delta f_n$ we define $\Delta g_n$ by (5). Then

$$
(\Delta g(I(k)))^d_{k=1} = A \xi^0
$$

and

$$
(\Delta g_n(I(k_1, \ldots, k_{n-1}, k)))^d_{k=1} = -h_{k_{n-1}}(k_1, \ldots, k_{n-1})A \xi^0, \quad \text{for } n \geq 2.
$$

Set $(x_k^0)_{k=1}^d = A \xi^0$. Then by (6)-(9),

$$
g_n(I(k_1, \ldots, k_n)) = \sum_{i=1}^d h_i(k_1, \ldots, k_n)x_i^0, \quad \text{for } n \geq 1.
$$

(For $n = 1$ this is clear from (6) and (8). Assume that this holds for $n - 1$. Then by (9) and (7),

$$
g_n(I(k_1, \ldots, k_n)) = g_{n-1}(I(k_1, \ldots, k_{n-1})) + \Delta g_n(I(k_1, \ldots, k_n))
$$

$$
= \sum_{i=1}^d h_i(k_1, \ldots, k_{n-1})x_i^0 - h_{k_{n-1}}(k_1, \ldots, k_{n-1})x_{k_{n-1}}^0
$$

$$
= \sum_{i=1}^d h_i(k_1, \ldots, k_n)x_i^0.
$$

Note that it follows from (7) that

$$
h_{k_{n-1}}(k_1, \ldots, k_{n-2}, k_{n-1}, k_{n-1}) = 0,
$$

$$
h_i(k_1, \ldots, k_{n-2}, k_{n-1}, k_{n-1}) = h_i(k_1, \ldots, k_{n-2}, k_{n-1}) \quad \text{if } i \neq k_{n-1},
$$

$$
h_i(k_1, \ldots, k_{n-2}, k_{n-1}, k_{n-1}, j) = h_i(k_1, \ldots, k_{n-2}, k_{n-1}, k_{n-1})
$$

for any $i, j \in \{1, \ldots, d\}$ and $n \geq 2$.

Thus, if $k_1 = k_2, k_3 = k_4, \ldots, k_{2d-1} = k_{2d}$, and if $\{k_1, k_3, \ldots, k_{2d-1}\} = \{1, 2, \ldots, d\}$, then

$$
h_i(k_1, k_2, \ldots, k_{2d-1}, k_{2d}) = 0 \quad \text{for any } i \in \{1, \ldots, d\}.
$$

Set

$$\mathcal{Q}_n = \{I(k_1, \ldots, k_n) : h_i(k_1, \ldots, k_n) = 0 \text{ for any } i \in \{1, \ldots, d\}\}.
$$

By (10),

$$\sum_{I \in \mathcal{Q}_{2d}} |I| \geq d^{2d} \cdot d!.
$$

Repeating this estimate, we get

$$\sum_{I \in \mathcal{Q}_{2d}} |I| \geq d^{2d} \cdot d! \left\{1 - \sum_{I \in \mathcal{Q}_{2(n-1)d}} |I|\right\} + \sum_{I \in \mathcal{Q}_{2(n-1)d}} |I|
$$

$$= d^{2d} \cdot d! + (1 - d^{2d} \cdot d!) \sum_{I \in \mathcal{Q}_{2(n-1)d}} |I|
$$

$$\geq d^{2d} \cdot d! \left\{1 + (1 - d^{2d} \cdot d!) + \cdots + (1 - d^{2d} \cdot d!)^{n-1}\right\}
$$

$$= 1 - (1 - d^{2d} \cdot d!)^n.$$
On the other hand, from (6)–(7), \(| h_k(k_1, \ldots, k_n) | \leq 2^n - 1. Thus,
\[
\| g_n \|_\infty \leq C_X 2^n - 1.
\]
Hence by (11)–(12),
\[
\int_{\Omega} |g_{2^nd}|^p \, dx = \int_{\{g_{2^nd} \neq 0\}} |g_{2^nd}|^p \, dx \leq (C_X 2^{2^nd})^p (1 - d^{-2d} \cdot d^1)^n \rightarrow 0 \quad \text{as } n \rightarrow \infty
\]
if \( p > 0 \) is small enough that
\[
2^{d^p}(1 - d^{-2d} \cdot d^1) < 1.
\]
But since \( f_n^*(x) \geq 1 \) on \( I(1) \),
\[
\int f_n^*(x)^p \, dx \geq d^{-1}.
\]
Thus,
\[
\| g_n \|_p / \| f_n^* \|_p \rightarrow 0 \quad \text{as } n \rightarrow \infty
\]
if (13) holds. This concludes the proof of Theorem 2.

**Remark.** In Chao [2], a conjecture about the best possible \( p_0 \) in Theorem A for the case \( d = 3 \) is given.

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**References**
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