INFINITE-DIMENSIONAL JACOBI MATRICES ASSOCIATED WITH JULIA SETS

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Abstract. Let B be the Julia set associated with the polynomial \( T_z = z^N + k_1z^{N-1} + \cdots + k_N \), and let \( \mu \) be the balanced \( T \)-invariant measure on \( B \). Assuming \( B \) is totally real, we give relations among the entries in the infinite-dimensional Jacobi matrix \( J \) whose spectral measure is \( \mu \). The specific example \( T_z = z^3 - \lambda z \) is given, and some of the asymptotic properties of the entries in \( J \) are presented.

1. Introduction. Let \( C \) be the complex plane and \( T: C \to C \) a polynomial, \( T(z) = z^N + k_1z^{N-1} + \cdots + k_N \) where \( N \geq 2 \) and each \( k_i \in C \). Define \( T^0(z) = z \) and \( T^n(z) = T \circ T^{n-1}(z) \) for \( n \in \{1, 2, 3, \ldots\} \). A fundamental role in the study of the sequence of iterates \( \{T^n(z)\} \) is played by the Julia set \( B \). \( B \) is the set of points \( z \in C \) where \( \{T^n(z)\} \) is not normal in the sense of Montel, and a general exposition can be found in Julia [8], Fatou [6, 7] and Brolin [5]. It has positive logarithmic capacity, and on it can be placed an equilibrium charge distribution \( \mu \). This provides a measure on \( B \) which is invariant under \( T: B \to B \) and is such that the system \( (B, \mu, T) \) is strongly mixing.

In an earlier paper [1] we investigated general properties of \( \mu \) and its associated orthogonal monic polynomials. Here we restrict attention to the case where \( B \) is a compact subset of the real line, and the orthogonal polynomials satisfy a three-term recurrence formula. In [2] we proved, for \( N = 2 \), relationships connecting the coefficients, which permit all the polynomials to be calculated in a recursive fashion. Here we generalized the relationships so that the orthogonal polynomials of all degrees can be obtained for any \( T \) for which \( B \) is a compact subset of the real line (Theorem 1). The results are illustrated for \( T(z) = z^3 - \lambda z \) with \( \lambda \geq 3 \). When \( \lambda = 3 \) the polynomials are those of Chebychev, shifted to the interval \([-2, 2]\), and when \( \lambda > 3 \) they become a generalization whose support is a Cantor set. In this case we establish that both the coefficients (Theorem 2) and the associated Jacobi matrix \( J \) (Theorem 3) display an asymptotic self-reproducing property.

2. Preliminaries.

Definition 1. \( \mu \) is a balanced \( T \)-invariant Borel measure on \( B \) if \( \mu \) is a probability measure supported on \( B \), such that for any complete assignment of branches of \( T^{-1} \), namely \( T_j^{-1} \) for \( j \in \{1, 2, 3, \ldots, N\} \), \( \mu(T_j^{-1}(S)) = \mu(S)/N \) for each Borel set \( S \).

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There is only one balanced $T$-invariant measure on $B$, and the equilibrium measure of Brolin is balanced [3]. If $\mu$ is balanced and $f \in L^1(B, \mu)$, then [1]

$\langle z^j f(T(z)) \rangle = S_j \langle f(z) \rangle / N$ for $j \in \{1, 2, \ldots, N - 1\},$

where $\langle f(z) \rangle = \int_B f(z) \, d\mu(z)$. Here

$$S_j = -j k_j - \sum_{l=1}^{j-1} k_l S_l$$

with $k_l$ the coefficient of $Z^{N-l}$ in $T$ for $l \in \{1, 2, \ldots, N\}$.

In [1] we showed that the sequence of monic polynomials $\{P_n(z)\}_{n=0}^{\infty}$, orthogonal with respect to $\mu$ according to $\langle P_l(z) P_m(z) \rangle = 0$ for $l \neq m$, obey the following relations:

(a) $P_1(z) = z + k_1/N$,
(b) $P_{mN}(z) = P_1(T(z))$ for $l \in \{0, 1, 2, \ldots\}$,
(c) $P_{N}(z) = T'(z) + k_1/N$ for $l \in \{0, 1, 2, \ldots\}$.

3. Results. When $B$ is a subset of the real line the orthonormal polynomials with respect to $\mu$ obey (b) and the following relation.

$$a(n + 1) p_{n+1}(x) + b(n) p_n(x) + a(n) p_{n-1}(x) = x p_n(x), \quad n \in \{0, 1, 2, \ldots\},$$

where

$$a(n) = \langle x p_n p_{n-1} \rangle \quad \text{for} \ n \in \{1, 2, 3, \ldots\},$$

and

$$b(n) = \langle x p_n^2 \rangle \quad \text{for} \ n \in \{0, 1, 2, \ldots\}.$$

The recurrence formula (3) can be recast as the formal operator equation

$$J \psi = x \psi$$

where

$$J = \begin{bmatrix} b(0) & a(1) & 0 & \cdots \\ a(1) & b(1) & a(2) & \cdots \\ 0 & a(2) & b(2) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and $\psi^T = (p_0, p_1, p_2, \ldots)$. $J$ can be treated as a selfadjoint operator acting on $l_2$. In [2] we showed that the coefficients in $J$ obey certain recurrence formulas when $T$ is quadratic; see also [4]. We generalize that result here.

Proceeding formally we have

$$J^l \psi = x^l \psi \quad \text{for} \ l \in \{0, 1, 2, \ldots\},$$

which leads to

$$\langle p_{mN} J^l \psi \varepsilon_{nN+1} \rangle = \langle x^l p_{mN}^2 \rangle$$
for \( n \in \{0, 1, 2, \ldots\} \), where \( \hat{e}_k \) is the \( l_2 \) vector with one in the \( k \)th place and zeros elsewhere. Observe also that the invariance of \( \mu \) together with (b) implies

\[
(8) \quad a(n) = \langle x^m p_{nN} p_{nN-N} \rangle = a(nN) a(nN-1) \cdots a(nN-N+1), \quad n \in \{1, 2, 3, \ldots\}.
\]

**Theorem 1.** Let \( a(n) = b(n-1) = 0 \) for \( n \leq 0 \). Then all of the coefficients in \( J \) can be calculated recursively using (8) and (7) with \( l \in \{1, 2, \ldots, 2N-1\} \).

The proof will require two lemmas.

**Lemma 1.** Let \( \{p_n\}_{n=0}^{\infty} \) be the orthonormal polynomials associated with the balanced \( T \)-invariant \( \mu \). Then

\[
(9) \quad \langle x^l p_{nN}^2 \rangle = D(l) \quad \text{for} \quad l \in \{1, 2, \ldots, 2N-1\},
\]

where

\[
D(l) = \begin{cases} 
N^{-1} S_l & \text{when} \quad l \in \{1, 2, \ldots, N-1\}, \\
N^{-1} S_{l-N} b(n) - \sum_{j=1}^{N} k_j D(l-j) & \text{when} \quad l \in \{N, \ldots, 2N-1\}, 
\end{cases}
\]

where \( S_0 = N \) and \( S_l \) is otherwise as defined in (2).

**Proof of Lemma 1.** For \( l \in \{1, 2, \ldots, N-1\} \) the result follows from (1) with \( f = p_{nN}^2 \). For \( l = N + m \),

\[
(10) \quad x^{N+m} = x^m T(x) - \sum_{j=1}^{N} k_j x^{m+N-j}.
\]

The lemma now follows on multiplying through by \( p_{nN}^2 \), integrating, and using the fact that

\[
(11) \quad \langle x^m T(x) p_{nN}^2 \rangle = N^{-1} S_m \langle x p_{nN}^2 \rangle = N^{-1} S_m b(n)
\]

for \( m \in \{0, 1, 2, \ldots, N-1\} \).

One can now see that the dependence on \( n \) on the right-hand side enters only through \( b(n) \).

**Lemma 2.** Let \( C_l(nN + 1, nN + 1) \) denote the \( (nN + 1, nN + 1) \) entry in \( J^l \). When \( l = 2k \), the coefficient in \( C^{2k}(nN + 1, nN + 1) \) with the highest index is \( a(nN + k) \) and all other coefficients have lower indices. When \( l = 2k + 1 \), the coefficients in \( C^{2k+1}(nN + 1, nN + 1) \) with the highest index are \( a(nN + k) \) and \( b(nN + k) \); all other coefficients have lower indices.

**Proof of Lemma 2.** We begin by computing \( C_l(nN + 1, nN + 1) \) with the aid of (7). Thus

\[
(12) \quad C_l(nN + 1, nN + 1) = a(nN) C^{l-1}(nN, nN + 1) + b(nN) C^{l-1}(nN + 1, nN + 1) + a(nN + 1) C^{l-1}(nN + 2, nN + 1), \quad l \in \{1, 2, \ldots, 2N-1\},
\]

with

\[
(13) \quad C^1(i, j) = a(i-1) \delta_{i-1,j} + b(i-1) \delta_{i,j} + a(i) \delta_{i+1,j},
\]
and

\[ C_m(i, j) = a(i - 1)C_{m-1}(i - 1, j)b(i - 1) + C_{m-1}(i, j) + a(i)C_{m-1}(i + 1, j). \]

It follows immediately from (14) that \( C_m(i, j) = 0 \) if \(|i - j| > m\). From (13) and (14) we find

\[ C^0(nN + 1, nN + 1) = b(nN), \]

and

\[ C^2(nN + 1, nN + 1) = a(nN)^2 + b(nN)^2 + a(nN) + 1)^2. \]

Let us now assume that the lemma holds up to \( 2k - 1 \). Then

\[ C^{2k}(nN + 1, nN + 1) = a(nN + 1)C^{2k-1}(nN + 2, nN + 1) + b(nN)C^{2k-1}(nN + 1, nN + 1). \]

One can easily show by induction that if \( a(l) \) or \( b(n) \) appear in \( C_m(i, j) \) then \( l \leq (m + i + j)/2 \) and \( n \leq (m + i + j - 1)/2 \). Consequently one need only consider the first term on the right-hand side of (17). Therefore

\[ C^{2k}(nN + 1, nN + 1) = \left[ \prod_{l=1}^{k} a(nN + l) \right] C^k(nN + k + 1, nN + 1) \]

\[ + \{ \text{terms containing only coefficients with indices lower than } nN + k \}. \]

But from (14) we have

\[ C^k(nN + k + 1, nN + 1) = \prod_{l=1}^{k} a(nN + l), \]

whence

\[ C^{2k}(nN + 1, nN + 1) = \left[ \prod_{l=1}^{k} a(nN + l) \right]^2 \]

\[ + \{ \text{terms involving only coefficients with indices lower than } nN + k \}. \]

Likewise,

\[ C^{2k+1}(nN + 1, nN + 1) = \left[ \prod_{l=1}^{k} a(nN + l) \right] C^{k+1}(nN + k + 1, nN + 1) \]

\[ + \{ \text{terms involving only } a(l) \text{ and } b(l - 1) \text{ with } l < nN + k \}, \]

and (14) now yields

\[ C^{2k+1}(nN + 1, nN + 1) = \left[ \prod_{l=1}^{k} a(nN + l) \right]^2 b(nN + k) \]

\[ + \{ \text{terms involving only } a(l) \text{ and } b(l - 1) \text{ with } l < nN + k \}. \]

This completes the proof of Lemma 2.
Proof of Theorem 1. If one is given \( a(i) \) and \( b(i) \) for \( i < Nn \), then Lemmas 1 and 2, together with (8), provide \( 2N \) relations from which one can explicitly calculate \( a(nN + l) \) and \( b(nN + l) \) for \( l \in \{0, 1, 2, \ldots, N - 1 \} \). This completes the proof.

Corollary 1. If \( B \) is an interval on the real line then \( B = [a, b] \) with \( a = -k_1/N - 2 \) and \( b = -k_1/N + 2 \). Moreover, \( d\mu = dx/\pi((b - x)(x - a))^{1/2} \), and \( T(x) + k_1/N \) is the monic Chebychev polynomial of degree \( N \) on \( B \).

Proof. If \( B \) is an interval then the electrical equilibrium distribution \( \mu \) is just the measure associated with the Chebychev polynomials of the first kind. Since all the off-diagonal entries in \( J \) except for \( a(1) \) are the same, (6) implies these must equal unity. Likewise, all diagonal entries in \( J \) must be equal to \(-k_1/N\), and the proof is completed.

4. An example. We examine the case \( T(z) = z^3 - \lambda z \) with \( \lambda \gg 3 \), for which Theorem 1 yields

\[
\begin{align*}
(22) & \quad b(n) = 0, \\
(23) & \quad a(3n + 1)^2 = 2\lambda/3 - a(3n)^2, \\
(24) & \quad a(3n + 2)^2 = \lambda/3
\end{align*}
\]

and

\[
\begin{align*}
(25) & \quad a(3n)a(3n - 1)a(3n - 2) = a(n).
\end{align*}
\]

From these relations and Corollary 1 it is easy to see that \( B = [-2, 2] \) when \( \lambda = 3 \). For \( \lambda > 3 \) it follows from [5] that \( B \) is a totally disconnected perfect subset of the real line, with Lebesgue measure zero. As such, it is a generalized Cantor set.

Lemma 3. For \( \lambda > 3 \) and \( n \in \{1, 2, 3, \ldots\} \), \( 0 < a(3n) < 1 \) and \( a(3n) < a(n) \).

Proof. From (23) and (25) it follows that \( a(1)^2 = 2\lambda/3 \) and \( a(3)^2 = 3/\lambda \). Furthermore, from (23)–(25) we have

\[
(26) \quad a(3n)^2 = \frac{3}{\lambda} \frac{a(n)^2}{2\lambda/3 - a(3n - 3)^2},
\]

and the lemma follows by induction and equations (23) and (24).

Theorem 2. For \( \lambda > 3 \) and \( m, s \in \{0, 1, 2, \ldots\} \),

\[
\text{Lim}_{n \to \infty} a(m3^n + s)^2 = a(s)^2.
\]

Proof. First consider the case \( s = 0 \). Then from (26)

\[
\begin{align*}
a(m3^n)^2 & = \frac{(3/\lambda)a(m3^{n-1})^2}{(2\lambda/3 - a(m3^n - 3)^2)} \\
& < \frac{(3/\lambda)a(m3^{n-1})^2}{(2\lambda/3 - 1)} < \frac{(3/\lambda)^n(2\lambda/3 - 1)^n a(m)^2}.
\end{align*}
\]

Because \( 3/\lambda < 1 \), and \( 2\lambda/3 - 1 > 1 \), for \( \lambda > 3 \) we now have \( \text{Lim}_{n \to \infty} a(m3^n)^2 = 0 \).

The proof is now completed by induction on \( m \) for \( s = 3m + k \), \( k \in \{0, 1, 2, \ldots\} \), using (23)–(25).
Results similar to Lemma 3 and Theorem 2 are valid for $T(z) = (z - \lambda)^2$ with $\lambda \geq 2$ and follow from [2]; see, for example, [4].

Now consider the sequence of infinite-dimensional Jacobi matrices $\{J^{(m\lambda^n)}\}$ defined for $m, n \in \{0, 1, 2, \ldots\}$ by

$$J^{(m\lambda^n)} = \begin{pmatrix}
0 & a(m\lambda^n + 1) & 0 \\
0 & a(m\lambda^n + 1) & a(m\lambda^n + 2) \\
a(m\lambda^n + 1) & 0 & a(m\lambda^n + 2) \\
a(m\lambda^n + 2) & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix}.$$  

Here the coefficients $a(i)$ are those determined by (23)–(25). Since the support $B$ of the spectral measure of $J$ is compact, it also is for each $J^{(m\lambda^n)}$, and, hence, each matrix corresponds to a selfadjoint operator in $l_2$.

**Theorem 3.** For each $m \in \{0, 1, 2, \ldots\}$ and $\lambda \geq 3$ the sequence of operators $\{J^{(m\lambda^n)}\}_{n=0}^{\infty}$ converges strongly to $J$.

This theorem, and indeed Theorem 2 also, are immediate when $\lambda = 3$ because then

$$J = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}.$$  

**Proof of Theorem 3.** Since the spectrum of $J$ is compact, the entries of $J^{(m\lambda^n)}$ are uniformly bounded. The result now follows since the weak convergence implied by Theorem 2 implies the strong operator convergence

$$\lim_{n \to \infty} \|J - J^{(m\lambda^n)}\| = 0, \quad \text{for all } x \in l_2,$$

for banded matrices. This completes the proof.

**References**

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