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THE STABILITY OF THE SINE EQUATION

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ABSTRACT. Let δ be a positive real constant and let G be an abelian group (written additively) in which division by 2 is uniquely performable. Every unbounded complex-valued function f on G satisfying the inequality

$$|f(x+y)f(x-y)-f(x)^2+f(y)^2|\leqslant\delta \quad \text{for all } x, y\in G$$

has to be a solution of the sine functional equation

$$f(x+y)f(x-y)=f(x)^2-f(y)^2 \quad \text{for all } x, y\in G.$$

A stability problem in functional equation theory seems to have been first raised by S. M. Ulam in [**4**] and D. H. Hyers who proved the stability of a linear functional equation (see [**3**]). Generalizations of this result appeared then in many papers. In 1979 J. A. Baker, J. Lawrence and F. Zorzitto showed in [**2**] that the equation $f(x+y)=f(x)f(y)$ has a different kind of stability. Exactly, they proved that if f is a real-valued function defined on a rational vector space V and satisfies the inequality $|f(x+y)-f(x)f(y)|\leqslant\delta$ for all $x, y\in V$ and some real $\delta>0$, then either f is bounded or $f(x+y)=f(x)f(y)$ for all $x, y\in V$. This result was then generalized by J. A. Baker (see [**1**]) who also had obtained a similar result for the cosine equation $f(x+y)+f(x-y)=2f(x)f(y)$. The main purpose of this paper is to prove an analogous result for the sine functional equation

$$f(x+y)f(x-y)=f(x)^2-f(y)^2. \tag{1}$$

We shall start from some general assumptions. Let G be an abelian group (written additively) in which division by 2 is uniquely performable (i.e. for each $x\in G$ there exists a unique $y\in G$ such that $y+y=x$; such a y will be denoted by $x/2$). Let f be a complex-valued function defined on G and such that the inequality

$$|f(x+y)f(x-y)-f(x)^2+f(y)^2|\leqslant\delta \tag{2}$$

holds for all $x, y\in G$ and some real $\delta>0$. Moreover, assume that f is an unbounded function.

For the proof we need the following three lemmas.

LEMMA 1.

$$f(0)=0. \tag{3}$$

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PROOF. Put $x = y$ and $u = 2x$ in inequality (2). Then

$$\delta \geqslant |f(u)f(0) - f(u/2)^2 + f(u/2)^2 = |f(u)|\,|f(0)|$$

and since $|f(u)|$ can be as large as desired, we must have

$$f(0) = 0. \quad \text{Q.E.D.}$$

Now put $x = (u+v)/2$, $y = (u-v)/2$ in (2). Then $x + y = u$, $x - y = v$ and inequality (2) assumes the form

$$\left| f(u)f(v) - f\left(\frac{u+v}{2}\right)^2 + f\left(\frac{u-v}{2}\right)^2 \right| \leqslant \delta \quad \text{for all } u, v \in G. \tag{4}$$

Since f is an unbounded function, there exists an $a \in G$ such that $|f(a)| \geqslant 4$. Let $g\colon G \to \mathbf{C}$ ($\mathbf{C}$ denotes here the field of all complex numbers) be the function defined by

$$g(x) = \frac{f(x+a) - f(x-a)}{2f(a)} \quad \text{for all } x \in G. \tag{5}$$

LEMMA 2. *For all $x, y \in G$ the following inequality is satisfied*:

$$|f(x+y) + f(x-y) - 2f(x)g(y)| \leqslant \delta. \tag{6}$$

PROOF. Let $x, y \in G$. Using (4) and (5) we get

$$\begin{aligned}
&|f(x+y) + f(x-y) - 2f(x)g(y)| \\
&\quad = \frac{1}{|f(a)|}|f(x+y)f(a) + f(x-y)f(a) - 2f(a)f(x)g(y)| \\
&\quad \leqslant \frac{1}{|f(a)|}\left| f(x+y)f(a) - f\left(\frac{x+y+a}{2}\right)^2 + f\left(\frac{x+y-a}{2}\right)^2 \right| \\
&\qquad + \frac{1}{|f(a)|}\left| f(x-y)f(a) - f\left(\frac{x-y+a}{2}\right)^2 + f\left(\frac{x-y-a}{2}\right)^2 \right| \\
&\qquad + \frac{1}{|f(a)|}\left| f\left(\frac{x+y+a}{2}\right)^2 - f\left(\frac{x-y-a}{2}\right)^2 - f(x)f(y+a) \right| \\
&\qquad + \frac{1}{|f(a)|}\left| f(x)f(y-a) - f\left(\frac{x+y-a}{2}\right)^2 + f\left(\frac{x-y+a}{2}\right)^2 \right| \\
&\qquad + \left| 2f(x)\frac{f(y+a) - f(y-a)}{2f(a)} - 2f(x)g(y) \right| \\
&\quad \leqslant \frac{4\delta}{|f(a)|} \leqslant \delta. \quad \text{Q.E.D.}
\end{aligned}$$

LEMMA 3. *The following equation is satisfied for all $x, y \in G$*:

$$f(x+y) + f(x-y) = 2f(x)g(y). \tag{7}$$

PROOF. Let x, y be two arbitrarily fixed points of G. Then, using (4) and (6), for all $z \in G$ we have

$$\begin{aligned}
&|f(z)|\,|f(x+y)+f(x-y)-2f(x)g(y)| \\
&\quad = |f(z)f(x+y)+f(z)f(x-y)-2f(x)f(z)g(y)| \\
&\quad \leqslant \left|f(z)f(x+y)-f\left(\frac{z+x+y}{2}\right)^2+f\left(\frac{z-x-y}{2}\right)^2\right| \\
&\qquad + \left|f(z)f(x-y)-f\left(\frac{z+x-y}{2}\right)^2+f\left(\frac{z-x+y}{2}\right)^2\right| \\
&\qquad + \left|f\left(\frac{z+x+y}{2}\right)^2-f\left(\frac{z-x+y}{2}\right)^2-f(z+y)f(x)\right| \\
&\qquad + \left|f\left(\frac{z+x-y}{2}\right)^2-f\left(\frac{z-x-y}{2}\right)^2-f(z-y)f(x)\right| \\
&\qquad + |[f(z+y)+f(z-y)]f(x)-2f(z)g(y)f(x)| \\
&\quad \leqslant 4\delta+\delta|f(x)| = \delta(4+|f(x)|).
\end{aligned}$$

Therefore

$$|f(x+y)+f(x-y)-2f(x)g(y)| \leqslant (4+|f(x)|)\delta/|f(z)|,$$

and since f is assumed to be an unbounded function and x is a fixed element, $(4+|f(x)|)\delta/|f(z)|$ is as small as we wish. Hence we must have

$$|f(x+y)+f(x-y)-2f(x)g(y)| = 0,$$

whence

$$f(x+y)+f(x-y) = 2f(x)g(y).$$

Since x and y were arbitrarily fixed elements of G, equation (7) holds for all $x, y \in G$. Q.E.D.

Now we are able to prove the main result of our paper.

THEOREM. *Every unbounded function $f\colon G \to \mathbf{C}$ satisfying inequality*

$$|f(x+y)f(x-y)-f(x)^2+f(y)^2| \leqslant \delta \quad \textit{for all } x, y \in G$$

has to be a solution of the equation

$$f(x+y)f(x-y) = f(x)^2 - f(y)^2.$$

PROOF. Let f be an unbounded solution of inequality (2). Put $x = 0$ in (7). Then, using (3), we obtain

$$f(y)+f(-y) = 0 \quad \text{for all } y \in G,$$

i.e.

$$f(y) = -f(-y) \quad \text{for all } y \in G. \tag{8}$$

Now, put $x = (u+v)/2$, $y = (u-v)/2$ in (7). Hence

$$f(u)+f(v) = 2f\left(\frac{u+v}{2}\right)g\left(\frac{u-v}{2}\right) \quad \text{for all } u, v \in G. \tag{9}$$

Then, from (3) and (9), we infer that

$$(10)\quad f(x+y)=f(x+y)+f(0)=2f\left(\frac{x+y}{2}\right)g\left(\frac{x+y}{2}\right)\quad \text{for all } x,\ y\in G,$$

and

$$(11)\quad f(x-y)=f(x-y)+f(0)=2f\left(\frac{x-y}{2}\right)g\left(\frac{x-y}{2}\right)\quad \text{for all } x,\ y\in G.$$

Using (8) and (9) we also have

$$(12)\quad f(x)-f(y)=f(x)+f(-y)=2f\left(\frac{x-y}{2}\right)g\left(\frac{x+y}{2}\right)\quad \text{for all } x,\ y\in G.$$

Now, using (9)–(12) we obtain

$$\begin{aligned} f(x+y)f(x-y) &= \left[2f\left(\frac{x+y}{2}\right)g\left(\frac{x+y}{2}\right)\right]\left[2f\left(\frac{x-y}{2}\right)g\left(\frac{x-y}{2}\right)\right] \\ &= \left[2f\left(\frac{x+y}{2}\right)g\left(\frac{x-y}{2}\right)\right]\left[2f\left(\frac{x-y}{2}\right)g\left(\frac{x+y}{2}\right)\right] \\ &= [f(x)+f(y)][f(x)-f(y)] = f(x)^2-f(y)^2 \end{aligned}$$

for all x, $y\in G$. Q.E.D.

REMARK 1. J. A. Baker proved in [**1**] that, concerning the stability of the cosine equation, a function g satisfying inequality

$$|g(x+y)+g(x-y)-2g(x)g(y)|\leqslant \delta$$

is either a solution of the cosine equation or it is bounded by a constant depending on δ only. The crucial step of the proof was that the function satisfying the above inequality is either bounded by such a constant or unbounded. It is not the case for the sine equation; indeed, the (bounded) functions

$$f_N(x)=N\sin x+1/N$$

satisfy inequality (2) with $\delta=3$,

$$|f_N(x+y)f_N(x-y)-f_N(x)^2+f_N(y)^2|\leqslant 3$$

for all real numbers x, y and all positive integers N. Nevertheless, for each positive real number M, the inequality $|f_N(x)|\leqslant M$ fails to hold for certain x and N.

REMARK 2. We must emphasize that our main result is proved for complex-valued functions only. Essential difficulties appear while considering functions with values in the algebra of quaternions or Cayley numbers, despite the multiplicativity of the norm; associativity, as well as commutativity, was necessary for us to prove our theorem.

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