THE STABILITY OF THE SINE EQUATION

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Abstract. Let $\delta$ be a positive real constant and let $G$ be an abelian group (written additively) in which division by 2 is uniquely performable. Every unbounded complex-valued function $f$ on $G$ satisfying the inequality

$$|f(x + y)f(x - y) - f(x)^2 + f(y)^2| \leq \delta$$

for all $x, y \in G$

has to be a solution of the sine functional equation

$$f(x + y)f(x - y) = f(x)^2 - f(y)^2$$

for all $x, y \in G$.

A stability problem in functional equation theory seems to have been first raised by S. M. Ulam in [4] and D. H. Hyers who proved the stability of a linear functional equation (see [3]). Generalizations of this result appeared then in many papers. In 1979 J. A. Baker, J. Lawrence and F. Zorzitto showed in [2] that the equation $f(x + y) = f(x)f(y)$ has a different kind of stability. Exactly, they proved that if $f$ is a real-valued function defined on a rational vector space $V$ and satisfies the inequality $|f(x + y) - f(x)f(y)| \leq \delta$ for all $x, y \in V$ and some real $\delta > 0$, then either $f$ is bounded or $f(x + y) = f(x)f(y)$ for all $x, y \in V$. This result was then generalized by J. A. Baker (see [1]) who also had obtained a similar result for the cosine equation $f(x + y) + f(x - y) = 2f(x)f(y)$. The main purpose of this paper is to prove an analogous result for the sine functional equation

(1) $$f(x + y)f(x - y) = f(x)^2 - f(y)^2.$$ 

We shall start from some general assumptions. Let $G$ be an abelian group (written additively) in which division by 2 is uniquely performable (i.e. for each $x \in G$ there exists a unique $y \in G$ such that $y + y = x$; such a $y$ will be denoted by $x/2$). Let $f$ be a complex-valued function defined on $G$ and such that the inequality

(2) $$|f(x + y)f(x - y) - f(x)^2 + f(y)^2| \leq \delta$$

holds for all $x, y \in G$ and some real $\delta > 0$. Moreover, assume that $f$ is an unbounded function.

For the proof we need the following three lemmas.

Lemma 1.

(3) $$f(0) = 0.$$
Proof. Put $x = y$ and $u = 2x$ in inequality (2). Then
\[ \delta \geq |f(u)f(0) - f(u/2)^2 + f(u/2)^2| = |f(u)| |f(0)| \]
and since $|f(u)|$ can be as large as desired, we must have
\[ f(0) = 0. \quad \text{Q.E.D.} \]
Now put $x = (u + v)/2$, $y = (u - v)/2$ in (2). Then $x + y = u$, $x - y = v$ and inequality (2) assumes the form
\[ (4) \quad |f(u)f(v) - f\left(\frac{u + v}{2}\right)^2 + f\left(\frac{u - v}{2}\right)^2| \leq \delta \quad \text{for all } u, v \in G. \]
Since $f$ is an unbounded function, there exists an $a \in G$ such that $|f(a)| \geq 4$. Let $g: G \to \mathbb{C}$ ($\mathbb{C}$ denotes here the field of all complex numbers) be the function defined by
\[ g(x) = \frac{f(x + a) - f(x - a)}{2f(a)} \quad \text{for all } x \in G. \]

Lemma 2. For all $x, y \in G$ the following inequality is satisfied:
\[ |f(x + y) + f(x - y) - 2f(x)g(y)| \leq \delta. \]

Proof. Let $x, y \in G$. Using (4) and (5) we get
\[ |f(x + y) + f(x - y) - 2f(x)g(y)| \]
\[ = \frac{1}{|f(a)|} |f(x + y)f(a) + f(x - y)f(a) - 2f(a)f(x)g(y)| \]
\[ \leq \frac{1}{|f(a)|} |f(x + y)f(a) - f\left(\frac{x + y + a}{2}\right)^2 + f\left(\frac{x + y - a}{2}\right)^2| \]
\[ + \frac{1}{|f(a)|} |f(x - y)f(a) - f\left(\frac{x - y + a}{2}\right)^2 + f\left(\frac{x - y - a}{2}\right)^2| \]
\[ + \frac{1}{|f(a)|} |f\left(\frac{x + y + a}{2}\right)^2 - f\left(\frac{x - y - a}{2}\right)^2 - f(x)f(y + a)| \]
\[ + \frac{1}{|f(a)|} |f\left(\frac{x + y - a}{2}\right)^2 + f\left(\frac{x - y + a}{2}\right)^2| \]
\[ + \frac{2f(x)f(y + a) - f(y - a) - 2f(x)g(y)}{2f(a)} - 2f(x)g(y) | \]
\[ \leq \frac{4\delta}{|f(a)|} \leq \delta. \quad \text{Q.E.D.} \]

Lemma 3. The following equation is satisfied for all $x, y \in G$:
\[ (7) \quad f(x + y) + f(x - y) = 2f(x)g(y). \]
Proof. Let $x, y$ be two arbitrarily fixed points of $G$. Then, using (4) and (6), for all $z \in G$ we have
\[
|f(z)| \left| f(x + y) + f(x - y) - 2f(x)g(y) \right| \\
\leq \left| f(z)f(x + y) + f(z)f(x - y) - 2f(x)f(z)g(y) \right| \\
\leq \left| f(z)f(x + y) - f\left( \frac{z + x + y}{2} \right)^2 + f\left( \frac{z - x - y}{2} \right)^2 \right| \\
+ \left| f(z)f(x - y) - f\left( \frac{z + x - y}{2} \right)^2 + f\left( \frac{z - x + y}{2} \right)^2 \right| \\
+ \left| f\left( \frac{z + x + y}{2} \right)^2 - f\left( \frac{z - x + y}{2} \right)^2 \right| - f(z + y)f(x) \\
+ \left| f\left( \frac{z + x - y}{2} \right)^2 - f\left( \frac{z - x - y}{2} \right)^2 \right| - f(z - y)f(x) \\
\leq 4\delta + \delta |f(x)| = \delta(4 + |f(x)|).
\]
Therefore
\[
|f(x + y) + f(x - y) - 2f(x)g(y)| \leq (4 + |f(x)|)\delta/|f(z)|,
\]
and since $f$ is assumed to be an unbounded function and $x$ is a fixed element, $(4 + |f(x)|)\delta/|f(z)|$ is as small as we wish. Hence we must have
\[
|f(x + y) + f(x - y) - 2f(x)g(y)| = 0,
\]
whence
\[
f(x + y) + f(x - y) = 2f(x)g(y).
\]
Since $x$ and $y$ were arbitrarily fixed elements of $G$, equation (7) holds for all $x, y \in G$. Q.E.D.

Now we are able to prove the main result of our paper.

Theorem. Every unbounded function $f : G \to \mathbb{C}$ satisfying inequality
\[
|f(x + y)f(x - y) - f(x)^2 + f(y)^2| \leq \delta \text{ for all } x, y \in G
\]
has to be a solution of the equation
\[
f(x + y)f(x - y) = f(x)^2 - f(y)^2.
\]

Proof. Let $f$ be an unbounded solution of inequality (2). Put $x = 0$ in (7). Then, using (3), we obtain
\[
f(y) + f(-y) = 0 \text{ for all } y \in G,
\]
i.e.
\[
f(y) = -f(-y) \text{ for all } y \in G.
\]
Now, put $x = (u + v)/2, y = (u - v)/2$ in (7). Hence
\[
f(u) + f(v) = 2f\left( \frac{u + v}{2} \right)g\left( \frac{u - v}{2} \right) \text{ for all } u, v \in G.
\]
Then, from (3) and (9), we infer that

\[(10) \quad f(x + y) = f(x + y) + f(0) = 2f\left(\frac{x+y}{2}\right)g\left(\frac{x+y}{2}\right) \quad \text{for all } x, y \in G, \]

and

\[(11) \quad f(x - y) = f(x - y) + f(0) = 2f\left(\frac{x-y}{2}\right)g\left(\frac{x-y}{2}\right) \quad \text{for all } x, y \in G.\]

Using (8) and (9) we also have

\[(12) \quad f(x) - f(y) = f(x) + f(-y) = 2f\left(\frac{x+y}{2}\right)g\left(\frac{x+y}{2}\right) \quad \text{for all } x, y \in G.\]

Now, using (9)–(12) we obtain

\[
f(x + y)f(x - y) = \left[2f\left(\frac{x+y}{2}\right)g\left(\frac{x+y}{2}\right)\right]\left[2f\left(\frac{x-y}{2}\right)g\left(\frac{x-y}{2}\right)\right]
\]

\[
= \left[2f\left(\frac{x+y}{2}\right)g\left(\frac{x+y}{2}\right)\right]\left[2f\left(\frac{x-y}{2}\right)g\left(\frac{x-y}{2}\right)\right]
\]

\[
= f(x) + f(y)\left[f(x) + f(y)\right] = f(x)^2 - f(y)^2
\]

for all \(x, y \in G\). Q.E.D.

Remark 1. J. A. Baker proved in [1] that, concerning the stability of the cosine equation, a function \(g\) satisfying inequality

\[|g(x + y) + g(x - y) - 2g(x)g(y)| \leq \delta\]

is either a solution of the cosine equation or it is bounded by a constant depending on \(\delta\) only. The crucial step of the proof was that the function satisfying the above inequality is either bounded by such a constant or unbounded. It is not the case for the sine equation; indeed, the (bounded) functions

\[f_N(x) = N \sin x + 1/N\]

satisfy inequality (2) with \(\delta = 3\),

\[|f_N(x + y)f_N(x - y) - f_N(x)^2 + f_N(y)^2| \leq 3\]

for all real numbers \(x, y\) and all positive integers \(N\). Nevertheless, for each positive real number \(M\), the inequality \(|f_N(x)| \leq M\) fails to hold for certain \(x\) and \(N\).

Remark 2. We must emphasize that our main result is proved for complex-valued functions only. Essential difficulties appear while considering functions with values in the algebra of quaternions or Cayley numbers, despite the multiplicativity of the norm; associativity, as well as commutativity, was necessary for us to prove our theorem.

References


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