ON FUNCTIONS THAT APPROXIMATE RELATIONS

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ABSTRACT. Let X be a metric space and let Y be a separable metric space. Suppose R is a relation in $X \times Y$. The following are equivalent: (a) for each $\varepsilon > 0$ there exists $f: X \to Y$ such that the Hausdorff distance from $f$ to $R$ is at most $\varepsilon$; (b) the domain of $R$ is a dense subset of $X$, and for each isolated point $x$ of the domain the vertical section of $R$ at $x$ is a singleton; (c) for each $\varepsilon > 0$ there exists $f: X \to Y$ of Baire class one such that the Hausdorff distance from $f$ to $R$ is at most $\varepsilon$.

Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. By a relation $R$ in $X \times Y$ we mean a nonempty subset of the product. Let us make $X \times Y$ a metric space by defining the distance $p$ between points $(x_1, y_1)$ and $(x_2, y_2)$ in the product by

$$p[(x_1, y_1), (x_2, y_2)] = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}.$$ 

A function $f: X \to Y$ will be said [3] to $\varepsilon$-approximate $R$ if each point in $f$ has $p$-distance at most $\varepsilon$ from some point in $R$, and each point of $R$ has $p$-distance at most $\varepsilon$ from some point in $f$. Alternatively, $f$ can be called [2] an $\varepsilon$-approximate selection for $R$, although this terminology has been used differently by Michael [7] and Deutsch and Kenderov [4]. More formally, if $f$ is an $\varepsilon$-approximate selection for $R$, then $f$ has Hausdorff distance at most $\varepsilon$ from $R$. We now pause to describe this notion.

Let $W$ be a metric space. For each point $w$ in $W$ let $S_{\varepsilon}[w]$ denote the open ball of radius $\varepsilon$ with center $w$ in $W$. If $C \subseteq W$ denote $\bigcup_{w \in C} S_{\varepsilon}[w]$ by $S_{\varepsilon}[C]$. If $K$ is another set in $W$ and there exists $\varepsilon > 0$ for which both $S_{\varepsilon}[C] \supseteq K$ and $S_{\varepsilon}[K] \supseteq C$, then the Hausdorff distance $\delta$ between $C$ and $K$ is given by

$$\delta[C, K] = \inf\{\varepsilon: S_{\varepsilon}[C] \supseteq K \text{ and } S_{\varepsilon}[K] \supseteq C\}.$$ 

If no such $\varepsilon$ exists, we write $\delta[C, K] = \infty$. Further information on this notion of distance can be found in Aubin [1], Kuratowski [6], or Nadler [9]. Now if $\delta$ denotes Hausdorff distance in $X \times Y$ as induced by $p$ and $R$ is a nonempty subset of $X \times Y$ and $f: X \to Y$, then the symbol $\delta[f, R]$ makes sense, and it is clear that (i) if $f$ $\varepsilon$-approximates $R$, then $\delta[f, R] \leq \varepsilon$; (ii) if $\delta[f, R] \leq \varepsilon$ then $f$ $\theta$-approximates $R$ for each $\theta > \varepsilon$.

The main purpose of this note is to characterize for arbitrary $X$ and separable $Y$ those relations in $X \times Y$ that admit for each $\varepsilon > 0$ a Borel $\varepsilon$-approximate selection. We shall in fact show that the existence for each $\varepsilon > 0$ of an $\varepsilon$-approximate selection
(Borel measurable or not) for the relation implies the existence for each $\epsilon > 0$ of a Baire class one $\epsilon$-approximate selection.

**Definition.** Let $X$ and $Y$ be metric spaces. A function $f: X \to Y$ is said to be of Baire class $\alpha < \Omega$ if for each open subset $G$ of $Y$ the set $f^{-1}(G)$ is of additive class $\alpha$ in $X$.

In particular, $f: X \to Y$ is of Baire class one if the inverse image of each open subset of $Y$ is an $F_\alpha$ subset of $X$. For a thorough discussion of such functions, the reader should consult Kuratowski [6], where the functions of Baire class $\alpha$ are called $B$-measurable of class $\alpha$. We need two results from this source, which we state as lemmas. The first is not deep; the second is a serious theorem of Montgomery [8].

**Lemma A.** Let $X$ and $Y$ be metric spaces. Suppose $\{A_i : i \in \mathbb{Z}^+\}$ is a collection of sets each of additive class $\alpha$ with union $X$. Suppose $f: X \to Y$ and for each $i \in \mathbb{Z}^+$ the restriction of $f$ to $A_i$ is of Baire class $\alpha$. Then $f$ is of Baire class $\alpha$.

**Lemma B.** Let $X$ be a metric space and let $F \subset X$. Suppose for each $x \in X$ there exists an open neighborhood $V_x$ of $x$ such that $F \cap V_x$ is of additive class $\alpha$. Then $F$ itself is of additive class $\alpha$.

Since open sets in a metric space are $F_\alpha$ sets, the phrase “$F \cap V_x$ is of additive class $\alpha$” used in Lemma B is unambiguous: subsets of $V_x$ that are of additive class $\alpha$ with respect to the relative topology on $V_x$ are precisely those that are of additive class $\alpha$ with respect to the topology on $X$. In the sequel we shall use the following notation for the domain and vertical section at $x$ of a relation $R$ in $X \times Y$:

$$\text{Dom}(R) = \{x : \text{for some } y, (x, y) \in R\}, \quad R(x) = \{y : (x, y) \in R\}.$$ 

**Theorem 1.** Let $X$ be a metric space and $Y$ a separable metric space. Suppose $R$ is a relation in $X \times Y$. The following are equivalent.

(a) For each $\epsilon > 0$ there exists $f: X \to Y$ such that $\delta[f, R] < \epsilon$.

(b) The domain of $R$ is a dense subset of $X$, and for each isolated point $x$ of the domain the section $R(x)$ is a singleton.

(c) For each $\epsilon > 0$ there exists $f: X \to Y$ of Baire class one such that $\delta[f, R] < \epsilon$.

**Proof.** (a) $\to$ (b). Suppose that $\text{Dom}(R) \neq X$. Then there exists $x \in X$ and $\epsilon > 0$ such that $S_x(x) \cap \text{Dom}(R) = \emptyset$. It follows that if $f: X \to Y$ is arbitrary, then $f \not\subset S_x[R]$, whence $\delta[f, R] \geq \epsilon$. Suppose now that $\text{Dom}(R) = X$, but for some isolated point $x$ of $\text{Dom}(R)$ the section $R(x)$ contains two distinct points $y_1$ and $y_2$ of $Y$. Since $x$ must be an isolated point of $X$, there exists $\epsilon > 0$ such that both $d_y(y_1, y_2) > \epsilon$ and $S_x(x) = \{x\}$. Hence if $f: X \to Y$ satisfies $\delta[f, R] < \epsilon/2$ we must simultaneously have $d_y(f(x), y_1) < \epsilon/2$ and $d_y(f(x), y_2) < \epsilon/2$, an impossibility.

(b) $\to$ (c). If $X$ has no limit points, then $\text{Dom}(R) = X$ and each vertical section of $R$ is a singleton. Thus, $R$ is a continuous function, and there is nothing to prove. Otherwise, let $\theta = \epsilon/2$ and let $L$ denote the set of limit points of $X$. Consider the family $\Omega$ of subsets $S$ of $L$ with the following property: for each $\{x, z\} \subset S$, $d_X(x, z) \geq \theta$. If $\Omega$ is partially ordered by inclusion, then by Zorn’s lemma $\Omega$ has a maximal member, say, $\{x_i : i \in I\}$, and it easily follows that $L \subset \cup_{i \in I} S_{x_i}[x_i]$. Let
$C = \{y_n: n \in \mathbb{Z}^+\}$ be a countable dense subset of $Y$, and for each $i \in I$ choose $y_{n(i)} \in C$ whose distance from $\bigcup \{R(x): x \in S_\theta[x_i] \cap \text{Dom}(R)\}$ is less than $\theta$. Since $W = \bigcup_{i \in I} S_\theta[x_i]$ as a subspace of $X$ is paracompact and regular, there is an open refinement $\{V_\lambda: \lambda \in \Lambda\}$ of the cover $\{S_\theta[x_i]: i \in I\}$ of $W$ such that $\{V_\lambda: \lambda \in \Lambda\}$ is a locally finite (closed) refinement of $\{S_\theta[x_i]: i \in I\}$. Let $E = \bigcup_{i \in I} S_\theta/3[x_i]$. We first define a Baire class one function on this open subspace of $X$. Let $i \in I$ be arbitrary. Since $x_i$ is a limit point of $X$, there is a sequence $\{x_{ni}\}$ of distinct points in $S_\theta/3[x_i]$ convergent to $x_i$. We define $h_i: S_\theta/3[x_i] \to Y$ as follows: let $h_i$ map $\{x_{ni}: n \in \mathbb{Z}^+\}$ onto a dense subset of $\bigcup \{R(x): x \in S_\theta[x_i] \cap \text{Dom}(R)\}$, and let $h_i$ assign to each remaining point of $S_\theta/3[x_i]$ the point $y_{n(i)}$. Functions with countable domains are automatically of Baire class one; so $h_i \mid \{x_{ni}: n \in \mathbb{Z}^+\}$ and $h_i \mid S_\theta/3[x_i] - \{x_{ni}: n \in \mathbb{Z}^+\}$ are both of Baire class one. Since the set $\{x_{ni}: n \in \mathbb{Z}^+\}$ and its complement in $S_\theta/3[x_i]$ are both $F_\sigma$ sets, Lemma A implies that $h_i$ itself is of Baire class one. Now set $h = \bigcup_{i \in I} h_i$. By the construction of $\{x_i: i \in I\}$, it is clear that $h$ is a well-defined function from $E$ to $Y$. Since the inverse image of each open set under $h$ is locally an $F_\sigma$ set, Lemma B ensures that $h$ is of Baire class one.

We next define a Baire class one function on $W - E$. For each $\lambda \in \Lambda$ choose $i(\lambda) \in I$ such that $\overline{V_\lambda} \subset S_\theta[x_{i(\lambda)}]$. For each $x \in W - E$ let $g(x) = y_{n(i(\lambda))}$ where $n(i(\lambda))$ is the smallest integer such that $x \in \overline{V_\lambda}$. We claim that for each $m \in \mathbb{Z}^+$ the set $g^{-1}((y_1, y_2, \ldots, y_m))$ is a relatively closed subset of $W - E$. To see this let $\{w_k\}$ be a sequence in $g^{-1}((y_1, y_2, \ldots, y_m))$ convergent to some point $w$ of $W - E$. Since $\{V_\lambda: \lambda \in \Lambda\}$ is locally finite there exist indices $\{\lambda_1, \lambda_2, \ldots, \lambda_p\} \subset \Lambda$ and an integer $K$ such that for each $k > K$, $\{\lambda: \lambda \in \Lambda$ and $w_k \in \overline{V_\lambda}\} \subset \{\lambda_1, \lambda_2, \ldots, \lambda_p\}$. Now for each $k > K$ there exists $\lambda(k) \in \{\lambda_1, \lambda_2, \ldots, \lambda_p\}$ such that $w_k \in \overline{V_{\lambda(k)}}$ and $n(i(\lambda(k))) \leq m$. Since $\{\lambda_1, \lambda_2, \ldots, \lambda_p\}$ is finite, there exists $\lambda \in \{\lambda_1, \lambda_2, \ldots, \lambda_p\}$ for which $w \in \overline{V_\lambda}$ and $n(i(\lambda)) \leq m$. This establishes the claim. Since the intersection of a closed set with an open set is an $F_\sigma$ set, for each $m \geq 2$,

$$g^{-1}((y_{m-1})) = g^{-1}((y_1, \ldots, y_m)) - g^{-1}((y_1, \ldots, y_{m-1}))$$

is a relatively $F_\sigma$ subset of $W - E$. Hence for each open set $G$ of $Y$ the set $g^{-1}(G) = g^{-1}(C \cap G)$ is a relatively $F_\sigma$ subset of $W - E$, and it follows that $g: W - E \to Y$ is of Baire class one.

On $X - W$ the relation $R$ reduces to a continuous function. Since the sets $E$, $W - E$, and $X - W$ are each $F_\sigma$ subsets of $X$, by Lemma A the function $f: X \to Y$ defined by

$$f(x) = \begin{cases} h(x) & \text{if } x \in E, \\
g(x) & \text{if } x \in W - E, \\
R(x) & \text{if } x \in X - W \end{cases}$$

is of Baire class one. It remains to show that $\delta[f, R] \leq \varepsilon = 2\theta$. We first show that each point in $f$ is within $\varepsilon$ of some point in $R$. If $x \in X - W$ then $(x, f(x)) \in R$. If $x \in W - E$ then there exists $i \in I$ such that $x \in S_\theta[x_i]$ and $f(x) = y_{n(i)}$. However, by the definition of $y_{n(i)}$ there exists a point $x_i^* \in S_\theta[x_i]$ and a point $y \in R(x_i^*)$ for

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which \( d_Y(y, y_{n(i)}) < \theta \). It follows that \( \rho[(x, f(x)), (x^*, y)] < \max\{2\theta, \theta\} = \epsilon \). Finally, if \( x \in E \) then there exists \( i \in I \) such that \( d_x(x, x_i) < \theta/3 \); moreover, \( f(x) \) is either \( y_{n(i)} \) or a point in \( \bigcup \{ R(z): z \in S_\theta[x_i] \cap \text{Dom}(R) \} \). In either case \((x, f(x))\) has \( \rho \)-distance less than \( \epsilon \) from some point \((x^*_i, y)\), where \( x^*_i \in S_\theta[x_i] \) and \( y \in R(x^*_i) \). We now must show that each point of \( R \) is within \( \epsilon \) of some point of \( f \). If \( x \in X - W \) then \( R(x) \) is a singleton and \( R(x) = f(x) \). Next let \( x \in W \cap \text{Dom}(R) \) and choose \( y \in R(x) \). There exists \( i \in I \) such that \( x \in S_\theta[x_i] \). Recall, however, that \( \{ f(x_n): n \in Z^+ \} \) is dense in \( \bigcup \{ R(z): z \in S_\theta[x_i] \cap \text{Dom}(R) \} \), so there exists \( n \in Z^+ \) for which \( d_Y(f(x_n), y) < \epsilon \). Again, it is clear that \( \rho[(x, y), (x_n, f(x_n))] < \epsilon \), and this portion of the proof is complete.

(c) \( \implies \) (a). Obvious.

Theorem 1 fails without the separability assumption on \( Y \).

**Example 1.** Let \( X \) be the rationals, viewed as a subspace of the line with the usual topology, and let \( Y \) be an uncountable set with the discrete metric. Let \( R = X \times Y \). Now each \( f: X \to Y \) has a countable range, and it follows from the definition of the metric \( \rho \) on \( X \times Y \) that \( \delta[f, R] = 1 \).

Following Michael we could call \( f: X \to Y \) an \( \epsilon \)-approximate selection for a relation \( R \) with domain \( X \) if, for each \( x \) in \( X \), \( f(x) \in S_\epsilon[R(x)] \). The existence of Baire class one approximate selections in this context would seem to rest on some continuity requirement on the map \( x \to R(x) \). For example, the property of *almost lower semicontinuity*, due to Deutsch and Kenderov [4], is sufficient [2]: for each \( x \) in \( X \) there exists a neighborhood \( V_x \) of \( x \) such that \( \bigcap \{ S_\epsilon[R(w)]: w \in V_x \} \) is nonempty.

**Example 2.** Let \( X = Y = [0,1] \) and let \( B \) be a non-Borel set in the interval. Let \( R \subset X \times Y \) be the characteristic function of the set \( B \). Then if \( f: X \to Y \) is a \( \frac{1}{2} \)-approximate selection for \( R \) (in the sense of Michael), then \( f^{-1}([\frac{1}{2}, 1]) = B \), a non-Borel set. Thus, \( R \) admits no Borel \( \frac{1}{2} \)-approximate selection.

Continuous approximate selections, either in our sense or that of Michael, can be obtained for certain well-behaved relations with convex vertical sections. A recent example: if \( X \) is paracompact and \( Y \) is a normed linear space, then those relations with domain \( X \) that admit for each \( \epsilon > 0 \) a continuous \( \epsilon \)-approximate selection in the sense of Michael are precisely those that are almost lower semicontinuous [4]. Invariably, such approximations are constructed by piecing together continuous functions defined locally via a partition of unity [5, p. 170] to yield a globally defined continuous function that is close to the relation. We close by showing that locally defined Baire class \( \alpha \) functions are subject to such an amalgamation, provided \( X \) is metric and \( Y \) is a second countable topological vector space.

**Theorem 2.** Let \( X \) be a metric space and let \( Y \) be a second countable topological vector space. Let \( \{ U_\lambda: \lambda \in \Lambda \} \) be a locally finite open cover of \( X \) and let \( \{ p_\lambda(\cdot): \lambda \in \Lambda \} \) be a partition of unity subordinated to the cover. Suppose for each \( \lambda \in \Lambda \) the function \( f_\lambda: U_\lambda \to Y \) is of Baire class \( \alpha \). Then \( f: X \to Y \) defined by \( f(x) = \sum_{\lambda \in \Lambda} p_\lambda(x)f_\lambda(x) \) is of Baire class \( \alpha \).

**Proof.** Fix \( x \) in \( X \) and let \( V_x \) be an open neighborhood of \( x \) that meets only finitely many members of the open cover, say \( \{ U_{\lambda_1}, \ldots, U_{\lambda_n} \} \). By Lemma B we need
only show that $f| V_x$ is of Baire class $\alpha$. Now for each $z$ in $V_x$ we have $f(z) = \sum_{i=1}^{\infty} p_{\lambda_i}(z) f_{\lambda_i}(z)$. Since the restriction of each function of Baire class $\alpha$ is of Baire class $\alpha$ on its restricted domain, to show that $f| V_x$ is of Baire class $\alpha$ it suffices to show that if $h_1: X \to Y$ and $h_2: X \to Y$ are of Baire class $\alpha$ and $p$ is a real valued continuous function on $X$, then both $ph_1$ and $h_1 + h_2$ are of Baire class $\alpha$. We prove the former statement, leaving the latter to the reader. Let $\{G_i: i \in Z^+\}$ and $\{U_i: i \in Z^+\}$ be bases for the topologies on $Y$ and the line, respectively. Consider $\phi: X \to Y \times R$ defined by $\phi(x) = (h_1(x), p(x))$. Since $\phi^{-1}(G_i \times U_j) = h_1^{-1}(G_i) \cap p^{-1}(U_j)$ and the sets of additive class $\alpha$ contain the open sets and are closed under finite intersections and countable unions, the second countability of $Y \times R$ implies $\phi^{-1}(G)$ is of additive class $\alpha$ for each open set $G$ in the product. Since $\psi: Y \times R \to Y$ defined by $\psi(y, \theta) = \theta y$ is continuous, $ph_1 = \psi \circ \phi$ is of Baire class $\alpha$.

It is important to note that Theorem 2 cannot be used to piece together locally Borel functions to obtain a globally Borel function. Using the well-known example of Szpilrajn-Marczewski [6] of a non-Borel set in a metric space that is nevertheless locally Borel, a counterexample can be easily constructed. The details are left to the reader.

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