PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 88, Number 4, August 1983

A NOTE ON THE STRONG MAXIMAL FUNCTION

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ABSTRACT. Given a nonnegative measurable function f on R^2 which is integrable over sets of finite measure, we construct a new function g with the same distribution function as f such that the strong maximal function of g has the same local integrability properties as its Hardy-Littlewood maximal function.

The Hardy-Littlewood maximal operator in R^n is defined by

$$Mf(x)=\sup\left\{\frac{1}{|Q|}\int_Q|f|: x\in Q\right\},$$

where Q is a cube in R^n with edges parallel to the coordinate axes. For f with bounded support, well-known arguments show that Mf is locally integrable provided $|f|\log_+|f|$ is integrable; E. M. Stein [**4**] proved that this condition on f is also necessary.

The strong maximal function $M_n f$ is defined in R^n similarly; the cubes are replaced by rectangles of arbitrary shape but oriented with edges parallel to the coordinate axes. Jessen, Marcinkiewicz, and Zygmund [**3**] noted $M_n f$ could be dominated by the composition of one-dimensional maximal operators; accordingly, $M_n f$ is integrable over sets of finite measure provided $|f|\,(\log_+|f|)^n$ is integrable, and no weaker local integrable condition on f is sufficient.

It has been conjectured by Fava, Gatto, and Gutiérrez [**2**] that this condition is also necessary; we became interested in this problem during the preparation of [**1**]. We show that the conjecture is false by constructing a rich class of functions on R^2 for which averages over rectangles are dominated by averages over squares. Consequently, local integrability properties of $M_2 f$ imply nothing stronger than the identical properties for Mf.

THEOREM. *Let f be a nonnegative measurable function on R^2 which is integrable over sets of finite measure. Then there is a function g on R^2 such that*

$$|\{x: f(x)>\lambda\}|=|\{x: g(x)>\lambda\}|,\qquad 0<\lambda<\infty,$$

and

$$M_2 g(x)\leqslant\frac{5}{4}Mg\left(\frac{x}{3}\right)+\sup_{|E|\leqslant 1}\int_E g,\quad a.e.\ x\in R^2.$$

Received by the editors November 16, 1982.
1980 *Mathematics Subject Classification*. Primary 42B25, 46E30.

PROOF. First we introduce some notation. We define

$$\begin{aligned} Q &= \{x \in R^2: |x_1| \leq 3 \text{ and } |x_2| \leq 3\}, \\ L_t &= \{x \in R^2: x_2 - x_1 = t\}, \\ D_t &= \{x \in R^2: |x_2 - x_1| \leq t\}, \\ A_t &= \{x \in D_t: |x_1 + x_2| \leq 2\}. \end{aligned}$$

For $x \notin L_0$, we let Q_x be the square with one corner at x and one diagonal along L_0. We define a function $t(\lambda)$, $0 < \lambda < \infty$, by

$$t(\lambda) = \tfrac{1}{4}|\{x \in R^2: f(x) > \lambda\}|$$

so that $|A_{t(\lambda)}| = |\{x: f(x) > \lambda\}|$. We now define our function g by

$$g(x) = \int_0^\infty \chi_\lambda(x)\, d\lambda,$$

where χ_λ is the characteristic function of $A_{t(\lambda)}$. It is then clear that the first condition on g is satisfied; we direct our attention to estimating $M_2 g$.

For the remainder of our argument, x will be a fixed point not in L_0 and R will be a generic rectangle containing x and having its sides parallel to the coordinate axes. We have

$$\int_R g = \int_0^\infty |A_{t(\lambda)} \cap R|\, d\lambda.$$

Choosing $\lambda_0 = \inf\{\lambda: |A_{t(\lambda)}| \leq 1\}$, we use $|A_{t(\lambda)} \cap R| \leq |R|$ for $\lambda < \lambda_0$ and show that for $|A_t| \leq 1$,

$$(*) \qquad 4|A_t \cap R|/|R| \leq |A_t| + 5|Q_x \cap 3A_t|/|Q_x|.$$

Assuming the validity of $(*)$,

$$\frac{1}{|R|}\int_R g \leq \lambda_0 + \frac{1}{4}\int_{\lambda_0}^\infty |A_{t(\lambda)}|\, d\lambda + \frac{5}{4|Q_x|}\int_{\lambda_0}^\infty |Q_x \cap 3A_{t(\lambda)}|\, d\lambda.$$

Since

$$\lambda_0 + \int_{\lambda_0}^\infty |A_{t(\lambda)}|\, d\lambda = \sup_{|E| \leq 1} \int_E g$$

and

$$\begin{aligned} \frac{1}{|Q_x|}\int_0^\infty |Q_x \cap 3A_{t(\lambda)}|\, d\lambda &= \frac{1}{|Q_{x/3}|}\int_0^\infty |Q_{x/3} \cap A_{t(\lambda)}|\, d\lambda \\ &= \frac{1}{|Q_{x/3}|}\int_{Q_{x/3}} g \leq Mg(x/3), \end{aligned}$$

this yields the desired conclusion.

We establish $(*)$ using elementary geometry. Calling $\mu_t E$ the one-dimensional measure of $E \cap L_t$, we first observe that $(\mu_0 R)/|R| \leq (\mu_0 Q_x)/|Q_x|$.

To see this, we note that we can change one side of R at a time until Q_x is reached without decreasing $(\mu_0 R)/|R|$ at any stage. Next, note that for $|t|$ small the ratio $(\mu_t R)/|R|$ is maximized by a square nearly the same as Q_x. Considering extreme cases shows that for $|3t| \leq |x_2 - x_1|$, we have $4(\mu_t R)/|R| \leq 9(\mu_t Q_x)/|Q_x|$ and, hence,

$$4|D_t \cap R|/|R| \leq 9|D_t \cap Q_x|/|Q_x|, \qquad 0 < 3t \leq |x_2 - x_1|.$$

For this last range of t, $|D_t \cap Q_x| \leq \frac{5}{9}|D_{3t} \cap Q_x|$, while for all larger t we have $Q_x \subset D_{3t}$. Consequently,

$$4|D_t \cap R|/|R| \leq 5|D_{3t} \cap Q_x|/|Q_x|, \qquad 0 < t < \infty.$$

For $x \in Q$ we have $Q_x \subset Q$ and $D_{3t} \cap Q = Q \cap 3A_t$; hence

$$4|A_t \cap R|/|R| \leq 4|D_t \cap R|/|R| \leq 5|D_{3t} \cap Q_x|/|Q_x| = 5|Q_x \cap 3A_t|/|Q_x|.$$

For $x \notin Q$ and $|A_t| \leq 1$ the analysis is trivial. Considering extreme cases shows every R meeting the complement of Q satisfies $4|A_t \cap R|/|R| \leq 4t = |A_t|$, and the proof is complete.

References

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