TALL $\alpha$-RECURSIVE STRUCTURES
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Abstract. The Scott rank of a structure $M$, $sr(M)$, is a useful measure of its model-theoretic complexity. Another useful invariant is $o(M)$, the ordinal height of the least admissible set above $M$, defined by Barwise. Nadel showed that $sr(M) \leq o(M)$ and defined $M$ to be tall if equality holds. For any admissible ordinal $\alpha$ there exists a tall structure $M$ such that $o(M) = \alpha$. We show that if $\alpha = \beta^+$, the least admissible ordinal greater than $\beta$, then $M$ can be chosen to have a $\beta$-recursive presentation. A natural example of such a structure is given when $\beta = \omega_1^\omega$ and then using similar ideas we compute the supremum of the levels at which $W(L_{\omega_1})$ singletons appear in $L$.

The results in this paper concern structures which are complicated model-theoretically, yet recursion-theoretically simple. Fix a structure $M$ for a language $\mathcal{L}$ of finite similarity type. The Scott rank of $M$ is defined as follows: Let $x, y, x', y'$ range over $|M|^{< \omega}$. By induction define a sequence of relations $\sim$ on members of $|M|^{< \omega}$ or the same length:

- $x_0 \sim y_0$ iff $x, y$ realize the same atomic type in $M$,
- $x_{\beta+1} \sim y_{\beta+1}$ iff $\forall x' \exists y' ((x \star x' \sim y \star y')$ and $\forall y' \exists x' (x \star x' \sim y \star y')$
- $x_\lambda \sim y_\lambda$ iff $x \sim y$ for all $\beta < \lambda, \lambda$ limit.

In the above, $\star$ denotes concatenation of sequences. Finally, Scott rank $(M)$ is the least $\alpha$ such that $\forall x \forall y (x_\alpha \sim y_\alpha \rightarrow x \sim y).$ Scott rank $(M)$ is a useful measure of the model-theoretic complexity of $M$.

Nadel [74] provides a bound on the Scott rank of a structure $M$ in terms of admissible set theory: Scott rank $(M) \leq o(M)$ where $o(M)$ is the ordinal height of the least admissible set above $M$ (see Barwise [69]). $M$ is tall if equality holds. This bound is best possible in that for any admissible ordinal $\alpha$ there is a tall structure $M$ such that Scott rank $(M) = \alpha$.

Let $\beta$ be a limit ordinal. $M$ is $\beta$-recursive if $|M| = \beta$ and all of the relations, functions of $M$, are $\beta$-recursive. (For a definition of $\beta$-recursive, see Friedman [78]. In this paper we need only consider those $\beta$ which are either admissible or the limit of admissible ordinals, in which case $\beta$-recursive coincides with $\Delta_1(L_\beta, \epsilon)$.) It is

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shown in Nadel [74] that there is an \( \omega \)-recursive (= recursive) structure of Scott rank \( \omega^1_1 \). (The example is a recursive linear ordering of ordertype \( \omega^\omega_1 + \omega^\omega_1 \cdot \eta \cdot \eta = \) ordertype of the rationals.) §1 of the present paper shows that for every limit ordinal \( \beta \) there is a \( \beta \)-recursive structure of Scott rank \( \beta^+ \), the least admissible ordinal greater than \( \beta \). Such a structure \( M_\beta \) is tall since it belongs to \( L_{\beta^+} \) and hence \( \omega(M) = \beta^+ \). Define \( \omega_{\omega_1} \)-rank \( (M) \) in exactly the same way as Scott rank \( (M) \) except where \( \bar{x}, \bar{y}, \bar{x}', \bar{y}' \ldots \) now range over \( |M|^{<\omega_1} \). §2 focuses on the special case: \( \beta = \omega_1 \). Using entirely different methods than in §1 a natural example of an \( \omega_1 \)-recursive structure of \( \omega_{\omega_1} \)-rank \( \omega^+_1 \) is presented (from this an \( \omega_1 \)-recursive structure of Scott rank \( \omega^+_1 \) is easily obtained). Similar techniques are then used to show that \( \Pi_1(L_{\omega_1}) \)-singletons appear cofinally inside \( L_\sigma \), where \( \sigma \) is the least stable ordinal greater than \( \omega_1 \).

1. Game rank versus Scott rank. The goal of this section is to prove

**Theorem 1.** For any limit ordinal \( \beta \) there is a \( \beta \)-recursive structure of Scott rank \( \beta^+ = \) least admissible ordinal greater than \( \beta \).

It clearly suffices to treat the case where \( \beta \) is either admissible or the limit of admissible ordinals. It will also be convenient to assume that \( \beta \) is greater than \( \omega \) (otherwise the result is known).

The proof of Theorem 1 can be outlined as follows: We first show that there is a \( \beta \)-recursive open game with a winning strategy for the “closed player”, but none inside \( L_{\beta^+} \). This allows one to build a \( \beta \)-recursive tree \( T \) of “game rank” \( \beta^+ \). Then a \( \beta \)-recursive structure \( M \) of Scott rank \( \beta^+ \) is obtained by building \( M \) so that its Scott analysis is very similar to the “game analysis” of \( T \).

We must first describe the “game rank” of a tree. All trees are subtrees of \( \beta^{<\omega} = \) all finite sequences of ordinals less than \( \beta \). Our definition here is rather nonstandard but is designed to allow the transition from game rank to Scott rank to go smoothly.

Let \( T \) be a tree. If \( \eta = (\eta(0), \eta(1), \ldots) \in T \) has even length we let \( \eta(\text{even}) = (\eta(0), \eta(2), \ldots) \). Let \( A_k = \{ \eta \in T | \eta \text{ has even length} \} \). For \( v \in A_k \) let \( B_v = \{ \eta \in T | \eta(\text{even}) = v \} \). If \( \eta \in T \) has even length we define \( R_k(\eta) \) by

\[
R_k(\eta) = 0 \iff \text{there is } v \supseteq (\eta(\text{even})) \text{ such that } \eta \text{ has no extension in } B_v, \quad \forall v \in \bigcup_k A_k \quad R_k(\eta) = \alpha > 0 \iff \exists \beta < \alpha \text{ such that } \forall \eta \supseteq (\eta(\text{even})) \text{ such that } \eta \in B_v, \eta \in \bigcup_k A_k \quad R_k(\eta) = \alpha \cdot 2^k
\]

Thus \( R_k(\eta) \) measures how good a position player I is in after \( \eta \) has been played in the following game: Players I and II alternately choose \( v_0, \eta_0, v_1, \eta_1, \ldots \) with the restrictions that \( v_0 \subseteq v_1 \subseteq \ldots, \eta_0 \subseteq \eta_1 \subseteq \ldots \in B_{v_i}, \quad v_i \in \bigcup_k A_k \). Player I wins if at some stage player II can make no legal move. Otherwise player II wins.

**Lemma 2.** There is a \( \beta \)-recursive tree \( T \) such that \( R_k(T) = \beta^+ \).

**Proof.** We use some ideas from \( \beta \)-logic. Enlarge the language of set theory by adjoining (Henkin) constants \( c_0, c_1, \ldots \) and a name \( \beta' \) for each ordinal \( \beta' \leq \beta \).
Formulas in this language can be easily coded by ordinals less than $\beta$. Let $S$ consist of the following sentences in this language:

(a) Axioms for admissibility,
(b) $\beta'$ is an ordinal, $\beta_1 \in \beta_2$ (whenever $\beta_1 < \beta_2 \leq \beta$).

Then the tree $T$ consists of all sequences of sentences $(\phi_0, \phi_1, \ldots)$ such that

(i) if $\phi_n = \psi$ then $\phi_{2n+1} = \psi$ or $\psi$,
(ii) if $\phi_{2n} = \exists x \psi$ then $\phi_{2n+1} = \psi(c_k)$ some $k$ or $\phi_{2n}$,
(iii) if $\phi_{2n} = \psi_1 \lor \psi_2$ then $\phi_{2n+1} = \psi_1$ or $\psi_2$ or $\phi_{2n}$,
(iv) if $\phi_{2n} = "c_k \in \beta"$ then $\phi_{2n+1} = "c_k = \beta"$ some $\beta' < \beta$ or $\phi_{2n}$,
(v) $\phi_1 \land \phi_3 \land \phi_5 \land \cdots$ is consistent with $S$.

Since $\beta > \omega$ condition (v) is $\beta$-recursive.

**Claim.** $Rk(T) = \beta^+$.

**Proof of Claim.** As the inductive definition of $Rk$ can be carried out in $L_{\beta^+}$ it is clear that $Rk(T) \leq \beta^+$. By absoluteness we can assume that $\beta$ is countable.

As $S$ has a model where $\beta$ is standard, $Rk(\phi) = \infty$. Now suppose $Rk(T) = \gamma < \beta^+$.

Let $\psi_0, \psi_1, \ldots$ be a listing of the sentences in this language. Define $\phi_0, \phi_1, \ldots$ by

$\phi_{2n} = \psi_n$, 
$\phi_{2n+1} = \text{least } \phi \text{ such that } (\phi_0, \ldots, \phi_n, \phi) \text{ has } \text{Rk} \geq \gamma$.

As $\{\eta \in T \mid \text{Rk}(\eta) \geq \gamma\} \in L_{\beta^+}$ the sequence $(\phi_0, \phi_1, \ldots) \in L_{\beta^+}$. But $\{\phi_{2n+1} \mid n \in \omega\}$ describes the complete Henkin theory of an end extension of $L_{\beta^+}$. This is a contradiction. Q.E.D.

We can now describe the structure $M$ to satisfy Theorem 1. Let $T$ be as in Lemma 2. Define $A_k, B_v$ for $v \in \bigcup_k A_k = A$ as before. Let $P_v = \{v \in A \mid \text{Endow each } P_v \text{ with a distinct } \nu_v \text{ so that } v_1 \neq v_2 \implies P_{v_1} \cap P_{v_2} = \emptyset \}$.

The universe of $M = \{|M| = \bigcup\{P_v \mid v \in A\}$: Introduce predicates for each $P_v$.

We now provide $P_v$ with an “affine” group structure; that is, a group structure without a distinguished identity. Note that $P_v$ is a group under the operation $\Delta$ of symmetric difference. For $w \in P_v$ let $S_{v,w} = \{\{w_1, w_2\} \mid w_1 \Delta w_2 = w\}$.

Notice that with these relations, any automorphism of $P_v$ is determined by its action at a single argument.

Finally, we introduce functions connecting the different $P_v$'s. If $v*(\alpha) \in A_n$ then $f_{v*(\alpha)}$ is defined by: $f_{v*(\alpha)}(w) = \{\eta \upharpoonright 2n - 2 \mid \eta \in w\}$ for $w \in P_{v*(\alpha)}$; $f_{v*(\alpha)}(w) = w$ otherwise. Thus any automorphism of $P_{v*(\alpha)}$ has a unique extension to $P_v$ preserving the function $f_{v*(\alpha)}$.

Thus the desired structure is $M = \langle |M|, P_v, S_{v,w}, f_{v*(\alpha)}, v \in A, w \in P_v \rangle$.

It remains to compute the Scott rank of $M$.

For any collection $G$ of partial functions from $M$ to $M$ define $G$-$Rk(g)$ for $g \in G$ by

$G$-$Rk(g) \geq 0 \iff g \in G$;
$G$-$Rk(g) \geq \alpha + 1 \iff \forall m \in |M| \exists h \in G(g \subseteq h, m \in \text{Dom}(h), G$-$Rk(h) > \alpha$) and
$\forall m \in |M| \exists h \in G(g \subseteq h, m \in \text{Range}(h), G$-$Rk(h) \geq \alpha$);
$G$-$Rk(g) \geq \lambda \iff \forall \alpha < \lambda \ G$-$Rk(g) \geq \alpha$ for limit $\lambda$;
$G$-$Rk(g) = \infty \iff G$-$Rk(g) \geq \alpha$ for all $\alpha$.

Also let $Rk(G) = \sup\{G$-$Rk(g) \mid g \in G, G$-$Rk(g) < \infty\}$. Thus we are interested in showing that $Rk(G_0) = \beta^+$ where $G_0$ is all finite partial isomorphisms of $M$.

For any $D \subseteq |M|$ let $\overline{D} = \text{closure } (D) = \bigcup\{P_v \mid \text{For some } v' \supseteq v, D \cap P_{v'} = \emptyset\}$.

As remarked earlier any partial isomorphism of $M$ with domain $D$ has a unique
extension to a partial isomorphism with domain (and range) \( D \). Thus it suffices to show that \( Rk(G_1) = \beta^+ \) where \( G_1 = \{ g \in G_0 \mid \text{Dom}(g) = \text{Dom}(g) \} \).

Now if \( g \in G_1 \) then \( g \) is uniquely determined by \( g^* \) which is defined by Domain \((g^*) = \{ v \mid P_v \subseteq \text{Dom}(g) \}, \ g^*(v) = g(\partial_v) \). Moreover, \( g^* \) satisfies

\[
(f_{v*}(\alpha)(g^*(v * (\alpha)))) = g^*(v).
\]

Conversely, any function \( h \) with domain a finite \( t \subseteq A \) closed under initial segments, obeying \((*)\) must be of the form \( g^* \) for some \( g \). Let \( H = \{ g^* \mid g \in G_1 \} \). Then \( Rk(G_1) = \text{Deg}(H) \) which is defined by

\[
\text{Deg}(h) \geq \alpha \iff h \in H; \\
\text{Deg}(h) \geq \alpha + 1 \iff \forall v \in A \exists h_1 \supseteq h(v \in \text{Dom}(h_1)), \text{Deg}(h_1) \geq \alpha; \\
\text{Deg}(h) \geq \lambda \iff \forall \alpha < \lambda \text{Deg}(h) \geq \alpha \text{ for limit } \lambda; \\
\text{Deg}(h) = \infty \iff \text{Deg}(h) \geq \alpha \text{ for all } \alpha, \text{Deg}(H) = \sup \{ \text{Deg}(h) \mid \text{Deg}(h) < \infty \}.
\]

Thus it suffices to show that \( \text{Deg}(H) = \beta^+ \).

Our final claim establishes the theorem by relating \( \text{Deg} \) (defined on \( H \)) to \( Rk \) (defined on \( \eta \in T, \text{length}(\eta) \) even).

**Claim.** For \( h \in H, \text{Deg}(H) = \min \{ Rk(\eta) \mid \eta \in h(v) \text{ for some } v \} \).

**Proof.** By induction on \( \alpha \) we show that \( \text{Deg}(h) \geq \alpha \iff Rk(h) \geq \alpha \iff Rk(\eta) \geq \alpha \) for all \( \eta \in \bigcup \text{Range}(h) \). This is trivial for \( \alpha = 0 \) or for limit \( \alpha \) (by induction). Let \( \alpha = \gamma + 1 \). Suppose \( Rk(\eta) \geq \gamma + 1 \) for all \( \eta \in \bigcup \text{Range}(h) \) and \( v \in A \). We show that \( \exists h_1 \supseteq h(v \in \text{Dom}(h_1)), \text{Rk}(\eta) \geq \gamma \) for all \( \eta \in \bigcup \text{Range}(h_1) \). Let \( v_0 \subseteq v \) be maximal, \( v_0 \subseteq \text{Dom}(h) \). For each \( \eta \in h(v_0) \) choose \( \eta' \supseteq \eta, \eta' \in B_{\lambda} \) so that \( \text{Rk}(\eta') \geq \gamma \) (this is possible since \( \text{Rk}(\eta) \geq \gamma + 1 \)). Then set \( h_1(v') = h(v') \) for \( v' \in \text{Dom}(h), h_1(v \uparrow k) = \{ \eta' \uparrow 2k \mid \eta \in h(v_0) \} \) for \( k \leq \text{length}(v) \).

Conversely suppose \( \text{Deg}(h) \geq \gamma + 1, \eta \in \bigcup \text{Range}(h) \). We show that for all \( v \supseteq (\eta)_{\text{even}} \), there is \( \eta' \supseteq \eta \) such that \( \eta' \in B_{\lambda}, \text{Rk}(\eta') \geq \gamma \). For, given \( v \supseteq (\eta)_{\text{even}} \), let \( h_1 \supseteq h, v \in \text{Dom}(h_1), \text{Deg}(h_1) \geq \gamma \). By induction, \( \text{Rk}(\eta') \geq \gamma \) for all \( \eta' \in h_1(v) \). But \( \eta \) has an extension \( \eta' \in h_1(v) \) as \( h_1 \in H \). Q.E.D.

Finally as \( Rk(T) = \beta^+ \) we conclude \( \text{Deg}(H) = \beta^+ \) and hence the theorem.

### 2. \( \omega_1 \)-recursive trees.

We use here Gödel condensation methods to build an \( \omega_1 \)-recursive tree \( T \) of \( L_{\omega_1 \omega_1} \)-rank \( \omega_1^+ \) = least admissible ordinal greater than \( \omega_1 \). For simplicity assume \( \omega_1 = \omega_1^+ \). The general case follows from the fact that the proof given below can be easily adapted to any \( L \)-cardinal \( \kappa \) such that \( \kappa \) is regular in \( L_\alpha \), \( \alpha = \text{least admissible greater than } \kappa \).

Let \( S = \{ \alpha < \omega_1 \mid \alpha \text{ admissible, } L_\alpha \models \omega_1 \text{ exists and is the largest admissible} \} \).

A typical member of \( S \) is \( \alpha \) where \( L_\alpha \) is the transitive collapse of a countable elementary submodel of \( L_{\omega_1^+} \).

We first define the tree \( T' = \{ (\alpha_0, \ldots, \alpha_n) \mid \text{For all } i, \alpha_i \in S, \alpha_i < \alpha_{i+1} \text{ and there exists } \Pi: L_\alpha \models L_{\alpha_{i+1}} \} \). Note that \( \Pi \) as above must be the identity on \( L_{\omega_1^+} \) and every element of \( L_\alpha \) is definable over \( L_{\alpha_i} \) from ordinals \( \leq \omega_1^{L_{\alpha_i}} \). Thus if \( \Pi \) exists in the definition of \( T' \) then \( \Pi^{-1} \) must be the transitive collapse of \( H = \text{Skolem hull of } L_{\alpha_{i+1}} \) inside \( L_{\alpha_{i+1}} \). This proves that \( T' \) is \( \omega_1 \)-recursive.

The desired tree \( T \) is obtained via a minor modification of \( T' \). This modification is needed to eliminate certain inhomogeneities on \( T' \): Define \( T = \{ ((\alpha_0, i_0), \ldots, (\alpha_n, i_n)) \mid \text{For all } k, \alpha_k \in S, i_k \in \omega, \alpha_k \leq \alpha_{k+1} \text{ and there exists } \Pi: L_{\alpha_k} \models L_{\alpha_{k+1}} \} \). Thus an
ordinal $\alpha \in S$ can be “repeated” countably often.) As before $T$ is $w_1$-recursive. Our goal is to show that $T$ has $L_{\omega_1 \omega_1}$-rank $w_1^{+}$. (We shall in fact show that $T$ is isomorphic to the tree $T$ in §1 of Friedman [81].)

We begin by analyzing the structure of $T$. We show that the structure of $T$ below $((\alpha_0, i_0), \ldots, (\alpha_n, i_n))$ is determined by the $S$-rank $(\alpha_n)$. This is defined by

$S$-rk$(\alpha) \geq 0 \iff \exists \alpha \in S$;

$S$-rk$(\alpha) \geq \gamma + 1 \iff$ For uncountably many $\alpha' \exists \Pi : L_\alpha \models L_{\alpha'}$, $S$-rk$(\alpha') \geq \gamma$;

$S$-rk$(\alpha) \geq \lambda \iff S$-rk$(\alpha) \geq \gamma$ for all $\gamma < \lambda$, for limit $\lambda$;

$S$-rk$(\alpha) = \infty \iff S$-rk$(\alpha) \geq \gamma$ for all $\gamma$.

Also set $\text{Rank}(S) = \sup\{ S\text{-rk}(\alpha) | \alpha \in S, S\text{-rk}(\alpha) < \infty \}$.

We can also define $\text{rk}((\alpha_0, i_0), \ldots, (\alpha_n, i_n)) = S$-rk$(\alpha_n)$, when $((\alpha_0, i_0), \ldots, (\alpha_n, i_n)) \in T$. Then a node on $T$ of rk 0 has exactly $\omega$-many immediate extensions on $T$. A node on $T$ of rk $\gamma > 0$ has exactly $\omega$-many immediate extensions of rk $\gamma$ and $\omega_1$-many immediate extensions of rk $\delta$ for $\delta < \gamma$. A node on $T$ of rk $\infty$ has $\omega_1$-many immediate extensions of rk $\infty$.

Our main goal is to show that for each $\sigma_0 \in T$, $\sigma_0 = 00$ or $\sigma_0 = \emptyset$, $\{ \sigma \in T | \sigma \supseteq \sigma_0 \text{ and } \text{rk} \sigma = 00 \}$ is $\omega$-many. From this it follows that $L_{\omega \omega_1}$-rank of $T = \omega_1^{+}$. Note that the inductive definition of rk as well as the inductive analysis of the $L_{\omega \omega_1}$-rank of $T$ can be carried out in $L_{\omega_1^{+}}$. If $\sigma_0 \in T$, $\text{rk} \sigma_0 = \infty$ then $\sigma_0$ must have immediate extensions of rk $\gamma$ for each $\gamma < \omega_1^{+}$ as otherwise $\{ \sigma \in T | \sigma \supseteq \sigma_0 \text{ and } \text{rk} \sigma = \infty \} = \{ \sigma \in T | \sigma \supseteq \sigma_0 \text{ and } \text{rk} \sigma = \gamma \}$ for some $\gamma < \omega_1^{+}$ and this latter set is a member of $L_{\omega_1^{+}}$. Thus we can conclude that if two nodes on $T$ lie on the same level and have the same rk, they can be mapped to each other by an automorphism of $T$. Thus determining the $L_{\omega \omega_1}$-type of nodes on $T$ is nothing more than determining their rk and the level of $T$ on which they lie. If $L_{\omega \omega_1}$-rank of $T$ is less than $\omega_1^{+}$ then $\{ \sigma \in T | \text{rk} \sigma = \infty \} = \{ \sigma \in T | \text{rk} \sigma = \gamma \}$ for some $\gamma < \omega_1^{+}$ and this latter set belongs to $L_{\omega_1^{+}}$. This contradicts our main claim.

**Claim.** $S$-rk$(\alpha) = \infty \iff \alpha < \omega_1$ and $\exists \Pi : L_\alpha \models L_{\omega_1^{+}}$.

From this claim it is clear that $\{ \sigma \in T | \sigma \supseteq \sigma_0, \text{rk} \sigma = \infty \} \notin L_{\omega_1^{+}}$ when $\text{rk} \sigma_0 = \infty$ or $\sigma_0 = \emptyset$, as otherwise $\{ \alpha < \omega_1 | \exists \Pi : L_\alpha \models L_{\omega_1^{+}} \} \in L_{\omega_1^{+}}$ which is impossible.

**Proof of Claim.** Clearly if $\alpha < \omega_1$ and $\exists \Pi : L_\alpha \models L_{\omega_1^{+}}$ then $S$-rk$(\alpha) = \infty$ as if $X$ is the set of all such $\alpha$'s then $X$ is uncountable and each element of $X$ can be elementarily embedded in all larger elements of $X$. For the converse suppose $\alpha \in S$, $S$-rk$(\alpha) = \infty$. Choose $\beta > \alpha$, $\exists \Pi : L_\alpha \models L_{\omega_1^{+}}$. Now inductively define $L_\alpha \models L_{\alpha_1} \models L_{\alpha_2} \models \cdots$ and $L_\beta \models L_{\beta_1} \models L_{\beta_2} \models \cdots$ such that $S$-rk $\alpha_i = \infty$ for each $i$ and $\beta_i < \alpha_i < \beta_{i+1}$. This is possible by the definition of S-rk. If Direct Lim$(L_{\alpha_i} | i < \omega) \models L_{\omega_1}$ is well-founded then it is isomorphic to some $L_{\alpha'}$. If Direct Lim$(L_{\beta_i} | i < \omega)$ is well-founded then it is isomorphic to some $L_{\beta'}$. But $\omega_1^{L_{\beta'}} = \omega_1^{L_{\alpha'}}$ so $\alpha' = \beta'$ since $\alpha', \beta' \in S$. We conclude that $\exists \Pi : L_\alpha \models L_{\alpha'}$, $\Pi_\beta : L_\beta \models L_{\alpha'}$, so $\Pi_\beta^1 \cup \Pi_\alpha : L_\alpha \models L_{\beta'}$ (since $\Pi_\alpha$, $\Pi_\beta$ is just the inverse of the transitive collapse of the Skolem hull of $\omega_1^{L_{\alpha'}}$, $\omega_1^{L_{\beta'}}$ in $L_\alpha$). So $\exists \Pi : L_\alpha \models L_{\omega_1^{+}}$.

It remains to justify the well-foundedness of the direct limits. This is provided by our final subclaim.

**Subclaim.** Direct Lim$(L_{\alpha_i} | i < \omega)$ is well-founded if $L_{\alpha_1} \models L_{\alpha_2} \models \cdots$ with $\alpha_1 < \alpha_2 < \cdots$ in $S$.  

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Proof. Let $M = \text{Direct Limit}(L_{\alpha_i} | i < \omega)$ and we identify $\text{sp}(M) =$ standard part of $M$ with some $L_\gamma$. Note that $\omega_1^M = \sup\{\omega_1^{L_{\alpha_i}} | i < \omega\} < \gamma$. But $\gamma$ is admissible as either $L_\gamma = M$ or $L_\gamma$ is the standard part of a model of $KP$. As $M \models \omega_1$ is the largest admissible, we can conclude that $\gamma = (\omega_1^M)^+$.

Now suppose $L_\gamma \neq M$ and choose $i$ and $\Pi: L_{\alpha_i} \rightarrow M$ so that $\text{Range}(\Pi) \not\subseteq L_\gamma$.

Let $\lambda < \alpha_i$ be so that $\Pi(\lambda) \not\in L_\gamma$. Then $\omega_1^{L_{\alpha_i}} < \lambda$. $L_{\alpha_i} \models \omega_1$ is the largest admissible, we may choose $\eta \in T$ such that $\text{Rk}(\eta) = \lambda$ where $T$ is the $\omega_1$-recursive tree constructed in Lemma 2 (where $\beta = \omega_1^{L_{\alpha_i}}$). Note that for arbitrary $\eta' \in T$, $\text{Rk}(\eta') < \infty$ if and only if player I has a winning strategy at position $\eta'$ for the game described immediately before Lemma 2.

If $T' =$ tree obtained from Lemma 2 when $\beta = \omega_1^M$ then $\Pi(T) = T'$ and $\Pi(\eta) = \eta$ has nonstandard $\text{Rk}' (= \text{Rk}$ for $T'$). But then player II has a winning strategy in the $T'$-game. This easily yields a winning strategy for player II in the $T$-game, contradicting $\text{Rk}(\eta) < \infty$. Q.E.D.

Thus we have established

**Theorem 3.** T is an $\omega_1$-recursive tree of $L_{\omega_1}$-rank $\omega_1^+$. 

An $\omega_1$-recursive structure of Scott rank $\omega_1^+$ can now be obtained by considering $T^\omega =$ infinite direct product of $\omega$-many copies of $T$. For then the analysis of $L_{\omega_1}$-rank for $T$ reduces to the Scott analysis of $T^\omega$.

We end with an observation concerning $\Pi_1(L_{\omega_1})$-singletons. Assume $V = L$. A function $f: L_{\omega_1} \rightarrow L_{\omega_1}$ is a $\Pi_1(L_{\omega_1})$-singleton if it is the unique solution to a $\Pi_1(L_{\omega_1})$ formula $\phi(f)$ with a single variable for a total function. An $\omega_1$-recursive tree with a unique branch of length $\omega_1$ yields a $\Pi_1(L_{\omega_1})$-singleton. We will show that for any $\beta < \sigma =$ least stable $\omega_1$ there is an $\omega_1$-recursive tree with a unique branch of length $\omega_1$ which is constructed in $L$ past $\beta$. Note that any $\Pi_1(L_{\omega_1})$-singleton must be a member of $L_{\omega_1}$.

Note that $L_\omega = \Sigma_1$ Skolem hull $(L_{\omega_1} \cup \{L_{\omega_1}\})$. Thus we can choose a $\Sigma_1$ formula $\phi(x,y,z)$ and $p \in L_{\omega_1}$ such that $\beta$ is the unique solution to $\phi(x,\omega_1,p)$. Let $\alpha$ be the least admissible such that $\beta < \alpha$, $L_{\alpha} \models \phi(\beta,\omega_1,p)$ and $\alpha^* = \Sigma_1$ projection of $\alpha = \omega_1$.

We describe now an $\omega_1$-recursive tree $T$ whose unique path $f$ consists of an $\omega_1$-sequence of elementary submodels of $L_{\alpha}$. This will suffice as clearly $f \not\in L_{\beta}$. $S$ consists of all $\alpha < \omega_1$ such that

(a) $L_{\alpha} \models KP + \omega_1$ exists, $\alpha^* = \omega_1^L_{\alpha}$;

(b) $p \in L_\gamma$ where $\gamma = \omega_1^L_{\alpha}$, $L_{\alpha} \models \phi(\beta,\gamma,p)$ for some $\beta < \alpha$;

(c) $L_{\alpha} \models$ There are no admissible $\delta > \beta$ s.t. $\delta^* = \omega_1$.

Then the tree $T = \{(\alpha_0,\alpha_1,\ldots) \in \omega^{<\omega_1} | \alpha_\delta \in S$ for all $\delta$, $\alpha_\delta =$ greatest $\alpha < \alpha_{\delta+1}$ s.t. $\exists \xi : L_{\alpha_\xi} \models L_{\alpha_{\delta+1}} \models \omega_1^{L_{\alpha_\delta}} = \bigcup\{\omega_1^{L_{\alpha_\delta}} \delta < \lambda\}$, $\lambda$ limit, $\sim \exists \xi < \alpha_0 \exists j : L_{\alpha_\xi} \models L_{\alpha_\xi} \models L_{\alpha_0}$. It is not hard to check that $T$ as above is uniquely determined as every element of $L_{\alpha}$ is definable over $L_{\alpha}$ from $\beta$ together with ordinals $\leq \omega_1^{L_{\alpha}}$, for $\alpha \in S$. So $T$ is $\omega_1$-recursive.

Now define an $\omega_1$-sequence of elementary submodels $M_0 < M_1 < \cdots$ of $L_{\alpha}$ by:

$M_0 =$ Skolem hull of $\{p,\omega_1,\beta\}$ in $L_{\alpha}$, $\gamma_0 = M_0 \cap \omega_1$; $M_{\delta+1} =$ Skolem hull of $\gamma_\delta \cup \{p,\omega_1,\beta\}$ inside $L_{\alpha_\delta} = M_{\delta+1} \cap \omega_1$; $M_\lambda = \bigcup\{M_\delta | \delta < \lambda\}$, $\gamma_\lambda = \bigcup\{\gamma_\delta | \delta < \lambda\}$ for limit $\lambda$. Then $\langle \alpha_0,\alpha_1,\ldots \rangle$ forms an $\omega_1$-branch through $T$ where $\alpha_\delta =$ transitive collapse ($M_\delta$).
If \( f \) is an \( \omega_1 \)-branch through \( T \) then there are elementary embeddings \( L_{f(0)} \cong L_{f(1)} \cong \cdots \) and we can form the direct limit \( L_\alpha'. \) Now \( \alpha' \) must be the least \( \mu \) such that \( \mu \) is admissible, \( \mu^* = \gamma', \mu > \beta', L_\mu \models \phi(\beta', \gamma', p) \) for some \( \beta' < \alpha', \gamma' = \omega_1^{L_\alpha'} \), but \( \gamma' = \omega_1 \). So \( \beta' = \beta \) since \( \beta \) is the unique solution to \( \phi(x, \omega_1, p) \). It follows that \( \alpha' = \alpha \) and hence \( f(\delta) = \alpha_\delta \) for all \( \delta \). Thus \( T \) has a unique \( \omega_1 \)-branch.

We have shown that \( \Pi_1(L_{\omega_1}) \)-singletons are constructed in \( L \) cofinally in the least stable ordinal \( \sigma > \omega_1 \). By way of contrast all \( \Pi_1(L_\omega) \)-singletons are constructed in \( L \) before \( \omega^+ = \omega_1^{CK} \). The disparity here is due to the fact that well-foundedness is easily expressible over \( L_{\omega_1} \).

**Final Note.** The second author has found a way to modify the construction in \( \S 1 \) to produce an \( \omega_1 \)-recursive structure of \( L_{\omega_1 \omega_1} \)-rank \( \omega_1^+ \). The key to the argument is in establishing the existence of an \( \omega_1 \)-recursive tree of \( \omega_1 \)-\( R_k \) where \( \omega_1 \)-\( R_k \) is defined in analogy to our earlier definition of \( R_k \). Then the appropriate structure is obtained from such a tree much as the structure \( M \) was obtained from \( T \) in \( \S 1 \).

**References**


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