IMMERSIONS OF REAL FLAG MANIFOLDS

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Abstract. This paper shows that, in many cases, Lam’s immersions of real flag manifolds are best possible.

1. Introduction. In [2], K. Y. Lam proved

Theorem. The real flag manifold

\[ RF(n_1, n_2, \ldots, n_s) = O(n_1 + n_2 + \cdots + n_s) / O(n_1) \times O(n_2) \times \cdots \times O(n_s) \]

can be immersed in Euclidean space with codimension \( \frac{1}{2} \sum n_i (n_i - 1) \), provided this codimension is nonzero.

The purpose of this note is to observe that this result is frequently best possible. Specifically, one has

Proposition. If \( \omega = \{n_1, \ldots, n_s\} \) can be partitioned as \( \omega = \omega_1 \cup \omega_2 \cup \omega_3 \) where \( \omega_1, \omega_2 \neq \emptyset \),

(1) \( \sum_{n \in \omega_1} n - \sum_{n \in \omega_2} n \leq 1 \), and

(2) \( m \in \omega_3 \) implies \( m \leq \sum_{n \in \omega_1} n + \sum_{n \in \omega_2} n + 1 \), then Lam’s result is best possible for \( RF(\omega) \).

For Grassmannians, \( RF(p, q) \), this result was proved by Hiller and myself [1], and the hypothesis is simply that \( q = p - 1 \), \( p \), or \( p + 1 \). In fact, the proof is a reduction to the Grassmannian case.

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2. The proof. According to the argument given by Lam [2, p. 313] the normal bundle of \( RF(n_1, \ldots, n_s) \) is realized by

\[ \eta = \lambda^2 \xi_1 + \lambda^2 \xi_2 + \cdots + \lambda^2 \xi_s, \]

where \( \xi_i \) is the standard \( n_i \)-plane bundle, and \( \lambda^2 \) denotes the second exterior power. The goal is to prove that the top-dimensional Stiefel-Whitney class of this bundle \( \eta \) is nonzero.

Lemma 1. If \( \omega \) refines \( \omega' \) and if the top normal Stiefel-Whitney class of \( RF(\omega') \) is nonzero, then the top normal Stiefel-Whitney class of \( RF(\omega) \) is nonzero.
Proof. Inductively, it suffices to consider \( \omega = \{n_1, n_2, \ldots, n_s\} \) and \( \omega' = \{n_1 + n_2, n_3, \ldots, n_t\} \) adding together two of the \( n_i \). One then has a fibration \( \pi : RF(\omega) \to RF(\omega') \) with fiber \( RF(n_1, n_2) \), with \( \pi^* \) monic in mod2 cohomology (since the cohomology injects into that of \( RF(1, \ldots, 1) \)). The map \( \pi \) splits the \((n_1 + n_2)\)-plane bundle over \( RF(\omega') \) into \( \xi_1 \oplus \xi_2 \) and hence

\[
\pi^*(\eta_\omega) = \lambda^2(\xi_1 \oplus \xi_2) \oplus \lambda^2\xi_3 \oplus \cdots \oplus \lambda^2\xi_s = (\xi_1 \oplus \xi_2) \oplus \eta_\omega.
\]

Thus \( w_{\text{top}}(\eta_\omega) : w_{\text{top}}(\xi_1 \oplus \xi_2) = \pi^*w_{\text{top}}(\eta_\omega) \neq 0 \) and so \( w_{\text{top}}(\eta_\omega) \neq 0 \). □

If one applies this lemma to the Proposition, adding together the integers in \( n_1 \) and the integers in \( n_2 \), one is reduced to consideration of \( RF(p, q, m_1, \ldots, m_t) \) where \( q = p - 1, p, \) or \( p + 1 \) and each \( m_i \leq p + q + 1 \).

Lemma 2. If \( n_1 \leq n_2 + \cdots + n_s + 1, s \geq 3, \) and the top normal Stiefel-Whitney class of \( RF(n_2, \ldots, n_s) \) is nonzero, then the top normal Stiefel-Whitney class of \( RF(n_1, n_2, \ldots, n_s) \) is nonzero.

Proof. Consider the map \( RF(1, \ldots, 1) \to RF(n_1, n_2, \ldots, n_s) \) so that

\[
w(\pi^*\xi_1) = \prod_{i=1}^{n_1} (1 + e_{i}), \ldots, w(\pi^*\xi_j) = \prod_{i=1}^{n_j} (1 + e_{n_1 + \cdots + n_{j-1} + 1}),
\]

writes each Stiefel-Whitney class in terms of one-dimensional classes. Here \( RF(1, \ldots, 1) = \text{Flag}(R^{n_1 + \cdots + n_s}) \), for which the mod 2 cohomology was described in [1]. One then has

\[
\pi^*w_{\text{top}}(\eta) = \prod_{i=1}^{s} s_{(n_i-1, \ldots, 1)}(e_{n_1 + \cdots + n_{i-1} + 1}, \ldots, e_{n_1 + \cdots + n_{i-1} + 1})
\]

by applying the lemma on p. 364 of [1]. Let \( m = n_2 + \cdots + n_s \), and

\[
a = e_1^{n_1 + m - 1} e_2^{n_2 + m - 3} \cdots e_j^{n_j + m - 2j + 1} \cdots e_{n_1}^{n_1 + m - 2n_1 + 1}.
\]

Then

\[
a \cdot \pi^*w_{\text{top}}(\eta) = e^{n_1 + m - 1} \cdots e_{n_1 + m - n_1} \left[ \prod_{i=2}^{s} s_{(n_i-1, \ldots, 1)}(e_{n_1 + \cdots + n_{i-1} + 1}) \right],
\]

which is nonzero if the second factor is nonzero in \( \text{Flag}(R^m) \). □

Applying this lemma to remove the integers \( m_i \) from \( \omega_3 \), one reduces the Proposition to the case \( RF(p, q) \) where \( q = p - 1, p, \) or \( p + 1 \). This in turn was proved in [1], to give the Proposition.

Comment. There are precisely four \( \omega = \{n_1, \ldots, n_s\} \) having each \( n_i \leq 4 \) to which the Proposition does not apply. They are \( \{3, 1\}, \{4, 1\}, \{4, 2\} \) and \( \{4, 1, 1\} \), and each has an improved immersion. \( RP^3 \) immerses in \( R^4 \), \( RP^4 \) in \( R^7 \), \( RF(4, 2) \) is an oriented 8-dimensional manifold with \( \tilde{w}_2\tilde{w}_6 = 0 \) and immerses in \( R^{14} \) by [3, Theorem (4.2.1)], while \( RF(4, 1, 1) \subset RP^3 \times RP^5 \) immerses in \( R^7 \times R^7 = R^{14} \). These improved immersions are best possible, since \( \tilde{w}_3, \tilde{w}_6 \) and \( \tilde{w}_5 \) are nonzero for the three latter cases.
REFERENCES


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