ON REGULAR-INVARIANCE OF CONTINUITY

R. G. ORI and M. RAJAGOPALAN

Abstract. Let \((X, \sigma)\) be a given topological space. A compression \(\tau\) of \(\sigma\) is regular-invariant if and only if for every regular space \(Y\) the \(\sigma\)-continuous functions into \(Y\) are also \(\tau\)-continuous. \(\sigma\) is regular minimal if no proper compression of \(\sigma\) is regular-invariant. J. A. Guthrie and H. E. Stone posed the problem of whether every semiregular space is regular minimal. We answer this question in the negative.

We begin by recalling a few definitions. Let \((X, \sigma)\) be a topological space. By a compression \(\tau\) of \(\sigma\) we mean a topology (on \(X\)) that is coarser than \(\sigma\). A compression \(\tau\) of \(\sigma\) is regular-invariant if for every regular space \(Y\) the \(\sigma\)-continuous functions into \(Y\) are also \(\tau\)-continuous. \(\sigma\) is regular minimal if no proper compression of \(\sigma\) is regular-invariant. Finally a topological space is semiregular if it has a base of regular open sets. Katetov [2] showed that every regular minimal space is semiregular. In [1] J. A. Guthrie and H. E. Stone asked if every semiregular space is regular minimal. We shall give an example to show that this is not true in general. Let \(X = [0, \infty) \times [0, \infty) - \{(0,0)\} \times (0,1)\), and let \(z\) denote the point \((0,0)\). We now define a topology on \(X\) by specifying a base for the neighborhoods of each point of \(X\): Let the neighborhood base of the point \(z\) consist of sets of the form \(\{z\} \cup (0, l/n) \times [0,1)\), \(n \in \mathbb{N}\). For every other point \(p \neq z\) let the neighborhood base consist of the sets \(S(p, l/n) \cap X, n \in \mathbb{N}\), where \(S(p, l/n)\) is the open disc (in \(R^2\) with the Euclidean topology) of radius \(1/n\) and centre \(p\). We denote the resulting topology on \(X\) by \(\sigma\). Clearly, \((X, \sigma)\) is a semiregular space.

We claim that \((X, \sigma)\) is not regular minimal: Firstly we define another topology on \(X\) by specifying the sets \(\{z\} \cup (0, l/n) \times [0,1 + 1/n)\), \(n \in \mathbb{N}\), to be a base for the neighborhoods of \(z\) while the neighborhood base for points \(p \neq z\) is the same as for \(\sigma\). Denote the resulting topology on \(X\) by \(\tau\). Note that \(\tau\) is a compression of \(\sigma\).

Now let \((Y, \mu)\) be a regular space and let \(f: (X, \sigma) \rightarrow (Y, \mu)\) be a continuous function. We claim that \(f\) is \(\tau\)-continuous.

Proof. It suffices to prove continuity of \(f\) at the point \(z\) only. Let \(V\) be an open subset of \(Y\) such that \(z \in f^{-1}(V)\). Since \((Y, \mu)\) is regular there exists an open subset \(W\) of \(Y\) such that \(f(z) \in W \subset \overline{W} \subset V\). Therefore \(z \in f^{-1}(W) \subset f^{-1}(\overline{W}) \subset f^{-1}(V)\). Since \(f\) is \(\sigma\)-continuous there exists a basic open subset \(B = \{z\} \cup (0, l/n) \times [0,1)\),
for some $n \in N$, such that $B \subset f^{-1}(W)$. It follows that
\[ \text{cl}_a(B) = \{z\} \cup (0, 1/n) \times [0, 1] \subset \text{cl}_a(f^{-1}(W)) \subset f^{-1}(\overline{W}) \subset f^{-1}(V). \]
In particular, we have that the set $C = [0, 1] \times \{1\} \subset f^{-1}(V)$. Therefore, for every point $p \in C$ there exists an integer $n(p) \in N$ such that $C \subset \bigcup_{p \in C} S(p, 1/n(p)) \subset f^{-1}(V)$. Since $C$ is compact in $(X, \sigma)$, there are a finite number of points $p_1, p_2, \ldots, p_m$ in $C$ such that
\[ C \subset \bigcup_{i=1}^{m} S(p_i, 1/n(p_i)). \]
Let $r = \max\{n, n(p_1), \ldots, n(p_m)\}$. It follows that
\[ \{z\} \cup (0, 1/r) \times [0, 1 + 1/r) \subset f^{-1}(V), \]
and this proves that $f$ is $\tau$-continuous at $(0, 0)$.

Acknowledgement. The main result of this paper has been announced much earlier (in 1977) by Jack Porter and P. L. N. Sarma (see [4]), though their results have not appeared in print so far. The authors thank the referee for bringing [4] to their attention.

REFERENCES


Department of Mathematics, University of Durban-Westville, Durban 4000, South Africa

Department of Mathematics, University of Toledo, Toledo, Ohio 43606

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use