LAGRANGE IDENTITY FOR POLYNOMIALS
AND δ-CODES OF LENGTHS 7t AND 13t

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Abstract. It is known that application of the Lagrange identity for polynomials (see [1]) is the key to composing four-symbol δ-codes of length (2s + 1)t for s = 2"10^626 + 1, and odd t ≤ 59 or t = 2^d10^626 + 1, where a, b, c, d, e and f are nonnegative integers. Applications of the Lagrange identity also lead to constructions of four-symbol δ-codes of length u for u = 7t or 13t. Consequently, new families of Hadamard matrices of orders 4uw and 20uw can be constructed, where w is the order of Williamson matrices. Related topics on zero correlation codes are also discussed.

1. Introduction. A sequence \( V = (v_k)_s = (v_1, v_2, \ldots, v_s) \) of vectors \( v_k \), where \( v_k \) is one of \( m \) orthonormal vectors \( i_1, i_2, \ldots, i_m \) or their negatives, is said to be an \( m \)-symbol δ-code of length \( s \) if \( v(j) = 0 \) for \( j \neq 0 \), where \( v(j) = \sum_{k=1}^{s} v_k \cdot v_{k+j} \) is the nonperiodic autocorrelation function of \( V \). Another characterization of \( V \) being an \( m \)-symbol δ-code of length \( s \) is that the associated polynomial

\[
V(z) = \sum_{k=1}^{s} v_k z^{k-1} = \sum_{j=1}^{m} p_j(z)i_j
\]

(where \( p_j(z) = \sum_{k=1}^{s} p_{jk} z^{k-1} \), \( \sum_{j=1}^{m} |p_{jk}| = 1 \) and all \( p_{jk} \in \{0, 1, -1\} \)) satisfy \( \sum_{j=1}^{m} |p_j(z)|^2 = s \), for any \( z \) in the unit circle \( K = \{z \in \mathbb{C}: |z| = 1\} \).

When \( m = 4 \) it is convenient to set \( i_1 = (1, 1, 0, 0) \), \( i_2 = (1, -1, 0, 0) \), \( i_3 = (0, 0, 1, 1) \) and \( i_4 = (0, 0, 1, -1) \) as four column vectors. All distinct \( i_j \) are obviously orthogonal to each other and have \( |i_j| = \sqrt{2} \) for each \( j \). Such a four-symbol δ-code of length \( s \) is called a regular δ-code of length \( s \) (abbreviated RD(s)). A sequence \( V = (v_k)_s \) of normalized vectors \( v_k \) is called a zero correlation code of length \( s \) if \( v(j) = 0 \) for \( j \neq 0 \). A regular zero correlation code of length \( s \) (abbreviated RZC(s)) is a zero correlation code of length \( s \) with vectors \( v_k = (q_k, r_k, s_k, t_k) \) such that \( |q_k| + |r_k| + |s_k| + |t_k| = 2 \) for each \( k \), where all \( q_k, r_k, s_k, \) and \( t_k \in \{0, 1, -1\} \). We note here that \( |v_k| = \sqrt{2} \) for each \( k \), and a regular δ-code is a regular zero correlation code in which all distinct vectors (excluding negatives) are orthogonal to each other.

The following theorem is proved in [1].

THEOREM 1 (LAGRANGE IDENTITY FOR POLYNOMIALS). Let \( a, b, c, d, e, f, g, h \) be polynomials of \( z \) in \( K \) with real coefficients. Also let

\[
W = -b'e + af' + cg + dh, \quad X = a'e + bf' + dg' - ch', \quad Y = -d'e - cf + ag' - bh, \quad Z = c'e - df + bg + ah',
\]

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where \( p' = p(z^{-1}) \) for \( p = p(z) \). Then
\[
|W|^2 + |X|^2 + |Y|^2 + |Z|^2 = (|a|^2 + |b|^2 + |c|^2 + |d|^2)(|e|^2 + |f|^2 + |g|^2 + |h|^2).
\]

Turyn base sequences for length \( t = 2m + p \) (abbreviated TBS(t)) are four \((1,-1)\)-sequences \((A, B; C, D)\), respectively, of lengths \( m + p \) and \( m \), pairwise, such that \( A = (a_k)_m \), \( B = (b_k)_m \), \( C = (c_k)_m \) and \( D = (d_k)_m \) satisfy \( a(j) + b(j) + c(j) + d(j) = 0 \) for \( j \neq 0 \).

TBS(t) exist for odd \( t \leq 59 \) or \( t = 2^a 10^b 26^c + 1 \), where \( a, b \) and \( c \) are nonnegative integers (see [4, 5]). We note here that, from given TBS(t), RD(t) can be found as follows: \((A, 0), (B, 0), (0, C) \) and \((0, D)\).

2. General results. We shall use \( \overline{P} = -P = (-p_n)_n \) and \( P' = (p_{n-k+1})_n \) to represent the negative and the reverse of a sequence \( P = (p_n)_n \). It should be noted that, as polynomials,
\[
P' = P(z^{-1}) = \sum_{k=1}^{n} p_k z^{-k} = z^{1-n} \sum_{k=1}^{n} p_{n-k+1} z^{k-1}
\]
for \( P = P(z) = \sum_{k=1}^{n} p_k z^k \).

Theorem 2. Let \((A, B; C, D)\) be given TBS(t). Then the following \((Q, R; S, T)\) is RD(t).
\begin{align*}
Q &= (q_k) = (\overline{A}, C; 0, 0; A, D; 0, 0; B', 0), \\
R &= (r_k) = (\overline{B}, D; 0, 0; B, \overline{C}; 0, 0; B, D; 0, 0; A', 0), \\
S &= (s_k) = (0, 0; A, \overline{C}; 0, 0; B, \overline{C}; 0, 0; A, C; 0, D'), \\
T &= (t_k) = (0, 0; B, D; 0, 0; A, \overline{D}; 0, 0; B, D; 0, C').
\end{align*}

Proof. In (2), each column vector \((q_k, r_k, s_k, t_k)\) is obviously one of \( i_1, i_2, i_3, i_4 \), or their negatives. Consequently, it suffices to prove that \( \sum_{k=1}^{n} |P(z)|^2 = 7t \) for any \( z \) in \( K \), where \( P_1 = (Q + R)/2, P_2 = (Q - R)/2, P_3 = (S + T)/2 \) and \( P_4 = (S - T)/2 \); or, equivalently, that
\[
|Q|^2 + |R|^2 + |S|^2 + |T|^2 = 14t \quad \text{for any } z \text{ in } K.
\]

First, we put \( a = A, b = B', c = -C, d = -D', e = 1, f = z^{-2t} + z^{2t}, g = (z^{-2t} + 1 - z^{2t})z^M \) and \( h = z^{4t-1} \), where \( M = t - m = m + p \), in (1). We can easily prove (2') from Theorem 1 by observing that \( Q = Yz^{2t+M}, R = Z'z^{2t+M}, S = Wz^{3t} \) and \( T = Xz^{3t} \);
\[
|a|^2 + |b|^2 + |c|^2 + |d|^2 = |A|^2 + |B|^2 + |C|^2 + |D|^2 = 2t,
\]
\[
|e|^2 + |h|^2 = 1 \quad \text{and} \quad |f|^2 + |g|^2 = 5 \quad \text{for any } z \in K.
\]

The above construction of RD(t) is based on the fact that there are nice sequences, \( e = (1), f = (1, 0, 1), g = (1, 1, -1) \) and \( h = (1) \), satisfying \( |e|^2 + |f|^2 + |g|^2 + |h|^2 = 7 \) for any \( z \) in \( K \); \( e(z) = e(z^{2t}) \) and \( f(z) = f(z^{2t}) \) are symmetric, i.e., \( e' = e(z^{-1}) = e \) and \( f' = f \); also \( e + f + g = g(z^{2t})z^Mz^{-2t} \) fit well to each other. Similarly Theorem 3 below is based on the fact that there are nice sequences,
e = (1, 0, 1, 0, 1), f = (−1, 0, 1, 0, −1, 1) and g = (−1, 1, 1, 1, 1, −1), satisfying |e|^2 + |f|^2 + |g|^2 = 13 for any z in K; g = g(z')z^{−5t/2} is symmetric; also, e, f (defined below) and g fit well.

**Theorem 3.** Let (A; B; C; D) be a given TBS(t). Then the following (Q; R; S; T) is RD(13t).

(3)

\[
Q = (A, D'; \bar{A}, C; A, D'; \bar{A}, C; A, D'; A, C; 0, 0; 0, 0; 0, 0; 0, 0; 0, 0),
\]

\[
R = (\bar{B}, C'; B, D; B, C'; B, D; \bar{B}, D; 0, \bar{D}; 0, 0; 0, 0; 0, 0; 0, 0; 0, 0),
\]

\[
S = (0, 0; 0, 0; 0, 0; 0, 0; 0, 0; 0, 0; 0, 0; 0, 0; 0, 0; 0, 0),
\]

\[
T = (0, 0; 0, 0; 0, 0; 0, 0; 0, 0; 0, 0; 0, 0; \bar{B}; \bar{D}; A'; D; A; D; \bar{B}; D; A'; D).
\]

**Proof.** In (3) it is obvious that each (q_k, r_k, s_k, t_k) is one of the i_j’s or their negatives. Therefore it is sufficient to prove that

\[
|Q|^2 + |R|^2 + |S|^2 + |T|^2 = 26t
\]

for any z in K.

We put a = A, b = B, c = C, d = D, e = (z^{−2t} + 1 + z^{2t})z^{t/2} = \text{expression involving } z, f = f(z')z'^{t/2} and g = g(z')z'^{−5t/2} in (1). We can prove (3) without difficulty from Theorem 1 by observing that Q = −Yz^{5t/2}, R = Zz^{5t/2}, S = −Wz^{19t/2 + M} and T = Xz^{19t/2 + M}, where M = t − m = m + p.

Golay complementary sequences of length s (abbreviated GCL(s)) are two (1,−1)-sequences of length s, f = (f_k)s and g = (g_k)s, having f(y') + g(y') = 0 for y =£ 0 or, equivalently, the associated polynomials satisfy |f|^2 + |g|^2 = 2s for any z in K (see [3]). The following theorem is somewhat similar to Theorem 2 of [1].

**Theorem 4.** Let (A; B; C; D) be TBS(t) and (f_k)s and (g_k)s GCL(s), where s > 1. Then the following (Q; R; S; T) is RZC((2s + 1)t).

(4)

\[
Q = (A_{f_1}, C_{g_1}; A_{f_{s−1}}, C_{g_{s−1}}; \ldots; A_{f_s}, C_{g_s}; 0, D; 0, 0; 0, 0; \ldots; 0, 0),
\]

\[
R = (A_{g_s}, \bar{C}_{f_1}; A_{g_{s−1}}, \bar{C}_{f_{s−1}}; \ldots; A_{g_1}, \bar{C}_{f_s}; \bar{B}, 0; 0, 0; 0, 0; \ldots; 0, 0),
\]

\[
S = (0, 0; 0, 0; \ldots; 0, 0; 0, 0; B_{f_1}, D_{g_1}; B_{f_{s−1}}, D_{g_{s−1}}; \ldots; B_{f_s}, D_{g_s}),
\]

\[
T = (0, 0; 0, 0; \ldots; 0, 0; A, 0; B_{g_1}, \bar{D}_{f_1}; B_{g_2}, \bar{D}_{f_2}; \ldots; B_{g_s}, \bar{D}_{f_s}).
\]

**Proof.** In (4) each (q_k, r_k, s_k, t_k) has \(\sqrt{2}\) as its norm. First we put a = A, b = B, c = C, d = D, e = 0,

\[
f = \sum_{k = −r}^{r−1} f_{r+k+1}z^{(2k+1)r/2 + M},
\]

\[
g = \sum_{k = −r}^{r−1} g_{r+k+1}z^{(2k+1)r/2},
\]

where r = s/2, and h = z^{(s+1)r/2} in (1). By observing that Q = Wz^{(s−1)r/2 + M}, R = Yz^{(s−1)r/2}, S = Xz^{(3s+1)r/2 + M} and T = Zz^{(3s+1)r/2}, and by noting that |A|^2 + |B|^2 + |C|^2 + |D|^2 = 2t, |f|^2 + |g|^2 = 2s and |h|^2 = 1 for any z in K, we obtain, from Theorem 1,

\[
|Q|^2 + |R|^2 + |S|^2 + |T|^2 = (|A|^2 + |B|^2 + |C|^2 + |D|^2)(|f|^2 + |g|^2 + |h|^2)
\]

\[
= 2t(2s + 1).
\]
Let

\[ (4') \quad Q = (A, 0), R = (B, 0), S = (0, C) \text{ and } T = (0, D) \] be \( \text{RD}(u) \), where \( A, B, C \) and \( D \) are \((0, \pm 1)\)-sequences such that \((U, V)\) is a \((-1, -1)\)-sequence for \( U = A \) or \( B \), and \( V = C \) or \( D \).\(^1\)

We note here that in Theorem 4, by replacing \( \text{TBS}(t): (A, B; C, D) \) with \( \text{RD}(u) \) of \((4')\) and \( t \) with \( u \) in \( \text{RZC} \), our argument is still valid. Consequently we have

**Theorem 4'.** Let \(((A, 0),(B, 0); (0, C),(0, D))\) be \( \text{RD}(u) \) as defined in \((4')\). Then \((4)\) is \( \text{RZC}((2s + 1)u) \).

Four \((1, -1)\)-sequences \((E, F, G, H)\) of length \( h \) are said to be complementary if \( e(j) + f(j) + g(j) + h(j) = 0 \) for \( j \neq 0 \) or, equivalently, if \(|E|^2 + |F|^2 + |G|^2 + |H|^2 = 4h\) for any \( z \) in \( K \). We obtain such complementary sequences \((E, F, G, H)\) from Theorem 4', where \( E = Q + T, F = Q - T, G = R + S \) and \( H = R - S \).

**Corollary 1.** There exist four complementary \((1, -1)\)-sequences of length

\[ h = (2s + 1)(2g + 1)t. \]

where \( s = 2^a10^b26^c \), \( g = 3, 6 \) or \( 2d10^e26^f \), and \( t \leq 59 \) or \( t = 2^f10^g26^h + 1 \); \( a, b, c, d, e, f, i, j, \) and \( k \) are nonnegative integers.

**Corollary 2.** There exist Goethals-Seidel (Hadamard) matrices of order \( 4h \), where \( h \) is defined in Corollary 1.

Although we have not found nice \((0, \pm 1)\)-sequences for composing \( \text{RD}(11t) \), \( \text{RZC}(11t) \) can be constructed as follows.

**Theorem 5.** Let \((A, B; C, D)\) be \( \text{TBS}(t) \). Then the following \((Q, R, S, T)\) is \( \text{RZC}(11t) \).

\[ (5) \]

\[
Q = (A, C; \tilde{A}, \tilde{C}; B', \tilde{C}, \tilde{A}, C; 0, D; 0, D; 0, D; 0, 0; 0, 0; 0, 0; 0, 0), \\
R = (A, \tilde{C}, A, D'; \tilde{A}, C, B, 0; B, 0; B, 0; 0, 0; 0, 0; 0, 0; 0, 0), \\
S = (0, 0; 0, 0; 0, 0; 0, 0; A, 0; A, 0; A, 0; B, D; \tilde{B}, C'; \tilde{B}, D, B, \tilde{D}), \\
T = (0, 0; 0, 0; 0, 0; 0, 0; 0, 0; 0, C, 0, C, 0, C, 0, C, \tilde{B}, \tilde{D}, B, D; A', D; B, D). \\
\]

**Proof.** In \((5)\), each \((q_k, r_k, s_k, t_k)\) has the same norm \( \sqrt{2} \). It is known that the four sequences \( e = (1), f = (-1, 0, -1, 1), g = (1, -1, -1, 1) \) and \( h = (1, 1, 1) \) satisfy \(|e|^2 + |f|^2 + |g|^2 + |h|^2 = 11\) for any \( z \) in \( K \). In \((1)\), we let \( e = z^{1/2-1}, f = f(z')z^{-3l/2}, g = g(z')z^{-3l/2} \) and \( h = h(z')z^{3l/2} \). Also let \( p = P \) for \( p = a, b, c \) and \( d \). By observing that \( Q = Wz^{3l/2+M}, R = -Yz^{3l/2}, S = Zz^{17t/2} \) and \( T = -Xz^{17t/2+M} \), we obtain from Theorem 1, \(|Q|^2 + |R|^2 + |S|^2 + |T|^2 = 22t\) for any \( z \) in \( K \).

From Theorem 5 we also obtain four complementary sequences of length \( 11t \):
\( E = Q + S, F = Q - S, G = R + T \) and \( H = R - T \).

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\(^1\)E.g. in \( \text{RD}(3r), (A, 0) = (A, C; 0, 0; \tilde{B}, 0), (B, 0) = (B, D; 0, 0; A', 0), (0, C) = (0, 0; A, \tilde{C}; 0, \tilde{D}) \) and \( (0, D) = (0, 0; B, \tilde{D}; 0, C') \); thus \( (A, C) = (A, C; A, \tilde{C}; B', \tilde{D}') \), etc.
Corollary 3. There exist four complementary \((1, -1)\)-sequences of length \(11t\).

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