SHORTER NOTES

The purpose of this department is to publish very short papers of unusually elegant and polished character, for which there is no other outlet.

A SHORT PROOF OF THE ALGEBRAIC WEIERSTRASS PREPARATION THEOREM

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Abstract. The contraction mapping principle yields a short proof of the algebraic Weierstrass Preparation Theorem.

Let $A$ be a complete local ring with maximal ideal $m$ [Z-S, p. 254]. We assume $A$ is separated in the $m$-adic topology, so $A \to \lim A/m^n$. We do not assume $A$ is noetherian. Let $B = A[[t]]$. A distinguished polynomial in $B$ is of the form $\pi = \pi_0 + \pi_1 t + \cdots + \pi_n t^n + \pi_{n+1} t^{n+1} + \cdots$, where $\pi_i \in m$. We shall give a proof of the

Weierstrass Preparation Theorem. If $f = \sum a_i t^i \in B$, where $a_i \in A$, and if $\exists n \in \mathbb{N}$ s.t. $a_i \in m$, $i < n$, and $a_n \notin m$, then $f = u\pi$, where $u$ is a unit of $B$ and $\pi$ is a distinguished polynomial of degree $n$. Also $u$ and $\pi$ are uniquely determined by $f$.

We make $A$ into a metric space [Z-S, p. 253] by setting

$$d(a, a') = \begin{cases} 0 & \text{if } a = a' \\ 2^{-s} & \text{if } a \neq a' \text{ and } a, a' \in m^s - m^{s+1} \end{cases}$$

Then $B$ becomes a metric space by the supremum of distance of coefficients: if $f = \sum a_i t^i$ and $f' = \sum a_i' t^i$, where $a_i, a_i' \in A$, then $d(f, f') = \sup_{i \in \mathbb{N}} d(a_i, a_i')$. It is readily checked that $B$ is a complete metric space. In addition, observe that if $b \in m[[t]]$, then $d(bf, bf') < \frac{1}{2} d(f, f')$.

Division Theorem. If $f, b \in B$ with $b \in m[[t]]$ and if $n \in \mathbb{N}$, then $f = q(t^n + b) + r$, where $q, r \in B$ and $r$ is a polynomial in $t$ of degree $< n$. In addition, $q$ and $r$ are uniquely determined by $f$ and $b$.

Proof. We remind the reader of the contraction mapping principle [K-K-O]: if $T: B \to B$ is a continuous map on a nonempty complete metric space $B$ s.t. $\exists 0 < k < 1$
with \( d(T(x), T(x')) \leq kd(x, x') \) for all \( x, x' \in B \), then \( \exists \) a unique \( q \in B \) s.t. \( T(q) = q \).

Define \( E: B \to B \) by \( x = p + E(x)t^n \), where \( p \) is a polynomial in \( t \) of degree \( < n \). Define \( T: B \to B \) by \( T(x) = E(f - xb) \); i.e. \( f - xb = p + T(x) \cdot t^n \), where \( p \) is a polynomial of degree \( < n \). If \( x' \in B \), then \( f - x'b = p' + T(x')t^n \), where \( p' \) is a polynomial of degree \( < n \), so subtracting we get

\[-b(x - x') = p'' + (T(x) - T(x')) \cdot t^n,
\]

where \( p'' \) is a polynomial of degree \( < n \). Observe that the coefficients of \( T(x) - T(x') \) involve only coefficients of the left-hand side of degree \( \geq n \). We deduce that

\[ d(T(x), T(x')) \leq d(bx, bx') \leq \frac{1}{2}d(x, x'), \]

since \( b \in m[[t]] \). Thus \( T \) is a contraction mapping, and we deduce from the contraction mapping principle that there is a unique \( q \in B \) such that \( T(q) = q \). That is, \( f - qb = r + qt^n \), or \( f = q(t^n + b) + r \), where \( r \) is a polynomial of degree \( < n \). This completes the proof of the division theorem.

The deduction of the preparation theorem from the division theorem is straightforward (cf. [B, p.41]).

**References**


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