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SHORTER NOTES

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A SHORT PROOF OF THE ALGEBRAIC WEIERSTRASS PREPARATION THEOREM

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ABSTRACT. The contraction mapping principle yields a short proof of the algebraic Weierstrass Preparation Theorem.

Let A be a complete local ring with maximal ideal $\mathfrak{m}$ [**Z-S**, p. 254]. We assume A is separated in the $\mathfrak{m}$-adic topology, so $A \xrightarrow{\sim} \varprojlim A/\mathfrak{m}^n$. We do not assume A is noetherian. Let $B = A[[t]]$. A distinguished polynomial in B is of the form $\pi = \pi_0 + \pi_1 t + \cdots + \pi_{n-1} t^{n-1} + t^n$, where $\pi_i \in \mathfrak{m}$. We shall give a proof of the

WEIERSTRASS PREPARATION THEOREM. *If $f = \sum_{i=0}^{\infty} a_i t^i \in B$, where $a_i \in A$, and if $\exists n \in \mathbf{N}$ s.t. $a_i \in \mathfrak{m}$, $i < n$, and $a_n \notin \mathfrak{m}$, then $f = u\pi$, where u is a unit of B and π is a distinguished polynomial of degree n. Also u and π are uniquely determined by f.*

We make A into a metric space [**Z-S**, p. 253] by setting

$$d(a, a') = \begin{cases} 0 & \text{if } a = a', \\ 2^{-s} & \text{where } a - a' \in \mathfrak{m}^s - \mathfrak{m}^{s+1} \text{ if } a \neq a'. \end{cases}$$

Then B becomes a metric space by the supremum of distance of coefficients: if $f = \sum a_i t^i$ and $f' = \sum a_i' t^i$, where $a_i, a_i' \in A$, then $d(f, f') = \sup_{i \in \mathbf{N}} d(a_i, a_i')$. It is readily checked that B is a complete metric space. In addition, observe that if $b \in \mathfrak{m}[[t]]$, then $d(bf, bf') \leqslant \frac{1}{2} d(f, f')$.

DIVISION THEOREM. *If $f, b \in B$ with $b \in \mathfrak{m}[[t]]$ and if $n \in \mathbf{N}$, then $f = q(t^n + b) + r$, where $q, r \in B$ and r is a polynomial in t of degree $< n$. In addition, q and r are uniquely determined by f and b.*

PROOF. We remind the reader of the contraction mapping principle [**K-K-O**]: if T: $B \to B$ is a continuous map on a nonempty complete metric space B s.t. $\exists\, 0 < k < 1$

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with $d(T(x), T(x')) \leqslant kd(x, x')$ for all $x, x' \in B$, then $\exists$ a unique $q \in B$ s.t. $T(q) = q$.

Define E: $B \to B$ by $x = p + E(x)t^n$, where p is a polynomial in t of degree $< n$. Define T: $B \to B$ by $T(x) = E(f - xb)$; i.e. $f - xb = p + T(x) \cdot t^n$, where p is a polynomial of degree $< n$. If $x' \in B$, then $f - x'b = p' + T(x')t^n$, where p' is a polynomial of degree $< n$, so subtracting we get

$$-b(x - x') = p'' + (T(x) - T(x')) \cdot t^n,$$

where p'' is a polynomial of degree $< n$. Observe that the coefficients of $T(x) - T(x')$ involve only coefficients of the left-hand side of degree $\geqslant n$. We deduce that

$$d(T(x), T(x')) \leqslant d(bx, bx') \leqslant \tfrac{1}{2}d(x, x'),$$

since $b \in \mathfrak{m}[[t]]$. Thus T is a contraction mapping, and we deduce from the contraction mapping principle that there is a unique $q \in B$ such that $T(q) = q$. That is, $f - qb = r + qt^n$, or $f = q(t^n + b) + r$, where r is a polynomial of degree $< n$. This completes the proof of the division theorem.

The deduction of the preparation theorem from the division theorem is straightforward (cf. [**B**, p.41]).

References

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