A NOTE ON CHARACTERS OF ALGEBRAIC GROUPS

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ABSTRACT. It is proven that a nowhere-vanishing polynomial function on a connected algebraic group, over an algebraically closed field, is a multiple of a (one-dimensional) character.

It is well known that any nowhere-vanishing polynomial function on GL_n(k), k algebraically closed, is a constant multiple of a power of the determinant. This is most easily seen by observing that the coordinate ring of GL_n(k) is k[ X_{ij}, \det^{-i}(X_{ij})] (i, j = 1, \ldots, n), so any unit in this ring, i.e., a nowhere-vanishing polynomial function f on GL_n(k), must be of the form f(x) = c \cdot \det^m(x), where c \in k^* and m is an integer. This result is not an isolated fact about GL_n(k); it holds for algebraic groups in general.

**Theorem.** Let G be a connected, affine, algebraic group over the algebraically closed field k, which may be of any characteristic. Let f: G \to k^* be any nowhere-vanishing polynomial function on G satisfying f(e) = 1. Then f is a character, i.e., f(xy) = f(x)f(y) for x, y \in G.

This theorem was originally proven by M. Rosenlicht [R, Theorem 3] using methods which are quite different from those used here. We shall use the well-known structure theory of affine algebraic groups (cf. J. E. Humphrey's monograph [H], for example) to give an elementary proof of this theorem. The author wishes to thank the referee for pointing out the article of M. Rosenlicht.

**Proof of the Theorem.** We prove the Theorem in four steps by first considering unipotent groups, then tori, solvable groups and, finally, the general case.

Case 1. Unipotent groups. Since G is unipotent, there is a descending chain of closed subgroups G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_s, such that G_{i+1} is normal in G_i, and G_i and the quotients G_i/G_{i+1}, i = 0, \ldots, s - 1, are one-dimensional unipotent groups. Any one-dimensional, connected, unipotent group is isomorphic to the additive group of k [H, Theorem 20.5]. Therefore, f, restricted to G_s, is a polynomial without zeros on k, so it must be a constant. As f(e) = 1 we see that f = 1 on G_s and is the trivial character. Each coset xG_s, x \in G_{s-1}, is isomorphic to k and the above argument shows that f is constant on xG_s as well. This implies there is a polynomial function f' \in k[G_{s-1}/G_s] such that f(x) = f'(\pi(x)), x \in G_{s-1}, where \pi: G_{s-1} \to G_{s-1}/G_s is the canonical projection. Because G_{s-1}/G_s is isomorphic to k, f' is also a

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constant, so \( f \) is the trivial character on \( G_{s-1} \). The remainder of the proof is a simple induction argument.

**Case 2. Tori.** If \( G = k^* \), then any polynomial function on \( G \) is of the form \( f(x) = x^n(a_0 + a_1 x + \cdots + a_m x^m) \), \( n, m \in \mathbb{Z}, \ m \geq 0, \ a_i \in k, \ a_0, a_m \neq 0 \). It is clear that, unless \( m = 0 \), \( f(x) \) will have a zero in \( k^* \), so \( f(x) = a_0 x^n \). However, \( a_0 = f(1) = 1 \), so \( f(x) = x^n \), a character of \( k^* \). For the general torus we argue by induction, so we assume the Theorem is true for any torus of dimension \( n \). An \((n + 1)\)-dimensional torus is of the form \( k^* \times T \), where \( T \) is an \( n \)-dimensional torus. For \( x \in k^* \), let \( f'_x \in k[T] \) be the function on \( T \) defined by \( f'_x(t) = f(xt)/f(x) \). Clearly, \( f'_x(e) = 1 \), and no \( f'_x \) vanishes on \( T \), so they are all characters of \( T \). There is a finite-dimensional vector subspace \( W \) of \( k[k^* \times T] \) containing \( f \) and invariant under the action of \( k^* \times T \) by left translations [H, Proposition 8.6]. If \( \rho: k[k^* \times T] \to k[T] \) is the projection given by restricting functions on \( k^* \times T \) to \( T \), then \( V = \rho(W) \) is finite dimensional and contains all \( f'_x, x \in k^* \). Since characters are linearly independent, \( V \) contains at most finitely many characters. The map \( k^* \to V, x \to f'_x \) is a morphism whose image is contained in a finite set and, since \( k^* \) is connected, its image must be a single point. This means \( f'_x = f'_e \) or \( f(xt) = f(x)f(t), x \in k^*, t \in T \), and this readily implies that \( f \) is a character since it is already a character when restricted to the subgroups \( k^* \) and \( T \).

**Case 3. \( G \) solvable.** The group \( G \) is a semidirect product \( TU \), where \( T \) is a maximal torus and \( U \) is the unipotent radical of \( G \) [H, Theorem 19.3]. By Cases 1 and 2, \( f \) is a character when restricted to \( T \) and is constant on the cosets \( tU, t \in T \). Using these facts, along with the normality of \( U \), a straightforward calculation shows that \( f \) is a character.

**Case 4. \( G \) any affine, connected group.** Let \( B \) be any Borel subgroup of \( G \). Again for \( g \in G \) we let \( f'_g \) be the function on \( B \) defined by \( f'_g(b) = f(gb)/f(g) \). By Case 3, each \( f'_g \) is a character of \( B \). From the proof of Case 2, we conclude that \( f(gb) = f(g)f(b) \) for \( g \in G, b \in B \). But, \( B \) was an arbitrary Borel subgroup of \( G \), and every element of \( G \) lies in some Borel subgroup of \( G \) [H, Theorem 22.2], so we conclude that \( f \) is a character of \( G \).

**References**


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