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GEODESICS AND JACOBI FIELDS IN BOUNDED HOMOGENEOUS DOMAINS

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ABSTRACT. We examine geodesics in bounded homogeneous domains equipped with an admissible Kähler metric. As in the symmetric case, there are no selfintersecting geodesics but, in contrast with the symmetric case, focal points can exist even in the Bergman metric.

1. Introduction. Let D be a Kähler manifold biholomorphic to a bounded homogeneous domain in a finite-dimensional complex vector space. Unless otherwise specified, we will assume the Kähler metric is a constant multiple of the Bergman metric. Since 1961 [**12**], it has been known that there exist nonsymmetric domains in which some sectional curvatures (even some holomorphic sectional curvatures) can be positive. Recently [**6**], it has been proved that D is symmetric if and only if all sectional curvatures are nonpositive. Thus it is natural to ask in what ways the existence of some positive curvature distinguishes the differential geometry or function theory of a nonsymmetric domain from that of a symmetric one.

Let b be a point in D and let γ be a unit speed geodesic with $\gamma(0) = b$. Two well-known consequences of nonpositive sectional curvature are that b has no conjugate points (i.e., any nontrivial Jacobi field along γ vanishing at b has no other zeros) and even no focal points (i.e., any nontrivial Jacobi field along γ vanishing at b has strictly increasing length, see [**9, 13**]). The first property says that b is a pole and allows one to introduce global geodesic polar coordinates which are useful in obtaining function-theoretic estimates, see [**11**]. The second property allows one to construct a boundary for D in terms of classes of asymptotic geodesics, see [**10**], also [**7**].

In this note, we are not able to answer the question about the existence of conjugate points although laborious calculations in many special cases (omitted) lead us to believe there are none. We will show that at least D has no selfintersecting geodesics. Perhaps more surprising, we will be able to show that there exist examples in which there are focal points. Our basic approach is to use normal j-algebras and we use the papers [**3–6**] as our general reference (although of course the whole subject originates in the work of Gindikin, Pyatetskii-Shapiro, and Vinberg (see e.g. [**8, 14, 15**])). The basic work of Gindikin, Pyatetskii-Shapiro, and Vinberg asserts that any bounded homogeneous domain with the Bergman metric is biholomorphically

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isometric to a Lie group with a left-invariant Kähler-Einstein metric and complex structure defined by a normal j-algebra structure on its Lie algebra.

2. Results. We now fix notation and assumptions for the rest of this paper. Let S be a connected, simply-connected Lie group with Lie algebra $\mathfrak{s}$ and suppose we are given $j\colon \mathfrak{s} \to \mathfrak{s}$, $\omega\colon \mathfrak{s} \to \mathbf{R}$ making $(\mathfrak{s}, j)$ into a normal j-algebra. Let S have the left-invariant Kähler metric defined by $\langle X, Y\rangle = \omega[jX, Y]$ (to be called an admissible metric and not necessarily the Bergman metric.) We have an orthogonal direct sum decomposition

$$\mathfrak{s} = \mathfrak{a} \oplus \sum_{\alpha} \mathfrak{n}_{\alpha}, \qquad \mathfrak{n} = \sum_{\alpha} \mathfrak{n}_{\alpha} = [\mathfrak{s}, \mathfrak{s}] \tag{1}$$

where $\mathfrak{a}$ is an abelian subalgebra, each α is a real linear functional on $\mathfrak{a}$ and $\mathfrak{n}_{\alpha}$ is the root space on which ad $\mathfrak{a}$ acts according to the root α, i.e.

$$\mathfrak{n}_{\alpha} = \{Y \in \mathfrak{s}\colon [H, Y] = \alpha(H)Y \text{ for all } H \in \mathfrak{a}\}.$$

There are linearly independent roots $\varepsilon_1, \ldots, \varepsilon_R$, $R = \dim \mathfrak{a}$, so that $\Sigma j\mathfrak{n}_{\varepsilon_k} = \mathfrak{a}$ and we may fix $X_k \in \mathfrak{n}_{\varepsilon_k}$ so that $\varepsilon_k(jX_l) = \delta_{kl}$. Every other root is of the form $\frac{1}{2}(\varepsilon_k \pm \varepsilon_l)$, $k < l$ or $\frac{1}{2}\varepsilon_k$ although not every such expression need be a root. We note that the covariant derivative ∇ of the Riemannian connection may be computed from

$$2\langle \nabla_X Y, Z\rangle = \langle [X, Y], Z\rangle + \langle [Z, X], Y\rangle + \langle [Z, Y], X\rangle, \qquad X, Y, Z \in \mathfrak{s}. \tag{2}$$

Pick an orthogonal basis $\{V_1, \ldots, V_N\}$ of $\mathfrak{s}$ so that $\{V_1, \ldots, V_R\} \subset \mathfrak{a}$, $\{V_{R+1}, \ldots, V_N\} \subset \mathfrak{n}$. Let γ be any geodesic. At any point $p \in S$, $\{(V_1)_p, \ldots, (V_N)_p\}$ is an orthogonal basis of T_pS so we may write

$$\gamma'(t) = \Sigma f_i(t)(V_i)_{\gamma(t)}. \tag{3}$$

For each t in the domain of γ, let $\tilde{\gamma}'(t)$ be the left-invariant field on S agreeing with $\gamma'(t)$ at the point $\gamma(t)$ so that

$$\tilde{\gamma}'(t) = \Sigma f_i(t)V_i. \tag{3$'$}$$

Then, again for fixed t, we have for any $W \in \mathfrak{s}$

$$\begin{aligned} 0 &= \langle \nabla_{\gamma'}\gamma', W\rangle(t) = \big(\langle \Sigma f_i' V_i, W\rangle + \Sigma f_i f_j \langle \nabla_{V_i} V_j, W\rangle\big)(t) \\ &= \big(\langle \Sigma f_i' V_i, W\rangle + \tfrac{1}{2}\Sigma f_i f_j\big(\langle [V_i, V_j], W\rangle + \langle [W, V_i], V_j\rangle + \langle [W, V_j], V_i\rangle\big)\big)(t) \\ &= \Sigma \langle V_i, W\rangle f_i'(t) + \langle [W, \tilde{\gamma}'(t)], \tilde{\gamma}'(t)\rangle. \end{aligned} \tag{4}$$

Define the element $H_0 \in \mathfrak{a}$ by $H_0 = \Sigma_{m=1}^{R} jX_m/m$. From the form of the possible roots of $\mathfrak{a}$ on $\mathfrak{n}$, we see that $\alpha(H_0) > 0$ for every root α. Clearly then $\alpha(H) > 0$ for all roots α and all H in a neighborhood $\mathfrak{a}_+$ of H_0 in $\mathfrak{a}$. Thus, for $H \in \mathfrak{a}_+$, ad H is a symmetric, positive semidefinite operator on $\mathfrak{s}$ which is positive definite on $\mathfrak{n}$.

We need the following lemma which is presumably well known but since we cannot find a specific reference, we will supply the proof.

LEMMA 1. *In a Riemannian homogeneous space, any nontrivial selfintersecting geodesic is periodic.*

PROOF. Suppose γ is a geodesic, parameterized by arclength, with $\gamma(0) = \gamma(\tau) = b$ for some fixed $\tau > 0$. We can find Killing vector fields $X_1^*, \ldots, X_N^*$ so that $\{(X_1^*)_b, \ldots, (X_N^*)_b\}$ is an orthonormal basis of the tangent space at b. Then $\gamma' \cdot \langle \gamma', X_k^* \rangle = \langle \gamma', \nabla_{\gamma'} X_k^* \rangle = 0$ so $\langle \gamma'(0), (X_k^*)_b \rangle = \langle \gamma'(\tau), (X_k^*)_b \rangle$. Thus $\gamma'(0) = \gamma'(\tau)$ which by uniqueness of the geodesic with prescribed initial vector shows that γ is periodic with τ a period.

Now we have enough information to prove our first main result.

THEOREM 1. *Let D be biholomorphically equivalent to a bounded homogeneous domain and be given an admissible Kähler metric corresponding to a simply-transitive, split-solvable group S of holomorphic automorphisms (in particular, this would include the Bergman metric). Then D has no nontrivial selfintersecting geodesics.*

PROOF. Suppose γ were a nontrivial periodic geodesic parameterized by arclength and write γ' as in (3). In (4), take $W = H \in \mathfrak{a}_+$. Then $\langle \operatorname{ad} H(\tilde{\gamma}'(t)), \tilde{\gamma}'(t) \rangle \geqslant 0$ implies $\Sigma \langle V_i, H \rangle f_i'(t) \leqslant 0$ for all t. Since the periodic function $\Sigma \langle V_i, H \rangle f_i$ then has nonpositive derivative, it must be constant and we have

$$0 = \Sigma \langle V_i, H \rangle f_i', \tag{5}$$

$$0 = \langle \operatorname{ad} H(\tilde{\gamma}'(t)), \tilde{\gamma}'(t) \rangle. \tag{6}$$

Since $\langle V_i, H \rangle = 0$ for $i > R$ and H can vary in the neighborhood $\mathfrak{a}_+$, (5) implies $f_i' = 0$ for $i \leqslant R$. Write $\tilde{\gamma}'(t) = (\tilde{\gamma}'(t))_{\mathfrak{a}} + (\tilde{\gamma}'(t))_{\mathfrak{n}}$, the sum of $\mathfrak{a}$ and $\mathfrak{n}$ components. Then (6) implies $0 = \langle \operatorname{ad} H((\tilde{\gamma}'(t))_{\mathfrak{n}}), (\tilde{\gamma}'(t))_{\mathfrak{n}} \rangle$. Since ad H is positive definite on $\mathfrak{n}$, we get $(\tilde{\gamma}'(t))_{\mathfrak{n}} = 0$. Thus $\gamma' = \Sigma_{i=1}^R a_i V_i \in \mathfrak{a}$ for some constants a_i. By homogeneity, we may assume $\gamma(0)$ is the identity of S so $\gamma(t) = \exp(t \Sigma_{i=1}^R a_i V_i)$. But for simply-connected, split-solvable Lie groups, it is known that exp: $\mathfrak{s} \to S$ is injective so γ cannot be periodic. By Lemma 1, this proves the theorem.

REMARK. The situation here is a special case of an admissible metric on an NC-algebra in the sense of Azencott-Wilson [**1**]. It would be interesting to see to what extent this theorem generalizes to the NC-algebra case. In this regard, we would like to thank J. Dorfmeister who simplified our original proof by suggesting the use of the operator ad H_0. The relevant point is that in any NC-algebra $\mathfrak{s}$, there exists an element H_0 in $\mathfrak{a}$ with $\alpha(H_0) > 0$ whenever $\alpha \neq 0$ and $\alpha + i\beta$ is a root.

Before proceeding, we want to rewrite (4) explicitly as a system of differential equations. Choose the orthogonal basis $\{V_1, \ldots, V_N\}$ so that each basis element is either in $\mathfrak{a}$ or in a root space. In (4), let $W = V_k$. Then the functions f_i satisfy the system of differential equations

$$0 = |V_k|^2 f_k' + \Sigma \langle [V_k, V_j], V_i \rangle f_i f_j, \qquad 1 \leqslant k \leqslant N, \tag{7}$$

where the coefficients $|V_k|^2$ and $\langle [V_k, V_j], V_i \rangle$ are constant. Note that $\langle [V_k, V_j], V_i \rangle = 0$ if $V_i \in \mathfrak{a}$ or if $V_i \in \mathfrak{n}_\alpha$, $V_j \in \mathfrak{n}_\beta$, $V_k \in \mathfrak{n}_\gamma$ with $\beta + \gamma \neq \alpha$ (here we use the convention that $\mathfrak{n}_0 = \mathfrak{a}$).

PROPOSITION 1. *Let α be any root, Y_α a unit vector in $\mathfrak{n}_\alpha$, $H_\alpha \in \mathfrak{a}$ the vector dual to α (i.e. $\langle H_\alpha, H \rangle = \alpha(H)$ for all $H \in \mathfrak{a}$). Let γ be a geodesic with*

$$\gamma'(0) = (f_0 H_\alpha + g_0 Y_\alpha)_{\gamma(0)}. \tag{8}$$

Then

$$\gamma'(t) = -\frac{c}{|H_\alpha|}\tanh(b + c|H_\alpha|t)H_\alpha + c\operatorname{sech}(b + c|H_\alpha|t)Y_\alpha, \tag{9}$$

where $c^2 = f_0^2|H_\alpha|^2 + g_0^2, f_0 = -c\tanh b/|H_\alpha|$, $g_0 = c\operatorname{sech} b$. (*These formulas are to be interpreted so that* γ' *is constant if* $\gamma'(0) \in \mathfrak{a}$.)

PROOF. Choose the basis $\{V_i\}$ so as to contain the vectors H_α, Y_α. It is easy to check that the functions f_k given by the coefficients in (9) satisfy the equations (7) with the appropriate initial conditions (note that in (7), $\langle [V_k, V_j], V_i\rangle f_i f_j = 0$ unless $V_i = Y_\alpha$ and $V_j = Y_\alpha$, $V_k = H_\alpha$ or $V_j = H_\alpha$, $V_k = Y_\alpha$).

Consider a fixed geodesic satisfying the initial condition (8). Without loss of generality, we may first normalize the metric so that $|H_\alpha| = 1$ (for this particular root α) and then we may normalize the parameter on the geodesic so that $|\gamma'|^2 = 1$. Then (9) becomes

$$\gamma'(t) = -\tanh(b + t)H_\alpha + \operatorname{sech}(b + t)Y_\alpha. \tag{9$'$}$$

PROPOSITION 2. *Let* α, β *be distinct roots and pick unit vectors* $Y_\alpha = \mathfrak{n}_\alpha$, $Z_\beta \in \mathfrak{n}_\beta$. *Suppose* $\nabla_{Y_\alpha} Z_\beta = [Y_\alpha, Z_\beta] = 0$. *Let* γ *be a geodesic with* $\gamma'(t)$ *given by* (9)$'$ (*after the appropriate normalization*). *The* fZ_β *is a Jacobi field along* γ *if and only if* $y = f(t)$ *satisfies the differential equation*

$$0 = y'' + \left\{-\langle H_\alpha, H_\beta\rangle \operatorname{sech}^2(b + t) - \left(\langle H_\alpha, H_\beta\rangle \tanh(b + t)\right)^2\right\} y. \tag{10}$$

Further, one always has $|\langle H_\alpha, H_\beta\rangle| \leqslant 1$.

PROOF. Note that $\nabla_H \equiv 0$ for any $H \in \mathfrak{a}$ (cf. [**4**, p. 407]) so one computes

$$R(H_\alpha, Z_\beta)H_\alpha = \langle H_\alpha, H_\beta\rangle^2 Z_\beta, \quad R(H_\alpha, Z_\beta)Y_\alpha = 0.$$

$$\begin{aligned} R(Y_\alpha, Z_\beta)H_\alpha &= \nabla_{Y_\alpha}\nabla_{Z_\beta}H_\alpha - \nabla_{Z_\beta}\nabla_{Y_\alpha}H_\alpha = \nabla_{Y_\alpha}[Z_\beta, H_\alpha] - \nabla_{Z_\beta}[Y_\alpha, H_\alpha] \\ &= -\beta(H_\alpha)\nabla_{Y_\alpha}Z_\beta + \alpha(H_\alpha)\nabla_{Z_\beta}Y_\alpha = 0. \end{aligned}$$

Also, $\nabla_{Y_\alpha}Y_\alpha = H_\alpha$ (cf. [**4**, p. 407]) so

$$R(Y_\alpha, Z_\beta)Y_\alpha = -\nabla_{Z_\beta}\nabla_{Y_\alpha}Y_\alpha = -\nabla_{Z_\beta}H_\alpha = [H_\alpha, Z_\beta] = \langle H_\alpha, H_\beta\rangle Z_\beta.$$

Thus

$$R(\gamma', fZ_\beta)\gamma' = f\left\{\langle H_\alpha, H_\beta\rangle^2 \tanh^2(b + t) + \langle H_\alpha, H_\beta\rangle \operatorname{sech}^2(b + t)\right\} Z_\beta.$$

Trivially, one finds $\nabla_{\gamma'}\nabla_{\gamma'}(fZ_\beta) = f''Z_\beta$. Thus the Jacobi equation $0 = \nabla_{\gamma'}\nabla_{\gamma'}(fZ_\beta) - R(\gamma', fZ_\beta)\gamma'$ is satisfied if and only if f satisfied (10).

The result that $|\langle H_\alpha, H_\beta\rangle| \leqslant 1$ follows from a direct case by case calculation allowing all possibilities for the roots α, β and the facts that $\langle X_k, X_k\rangle = \omega(X_k)$, $H_{\varepsilon_k} = jX_k/\omega(X_k)$ (see [**3**, p. 63]). Thus for example, one computes

$$\left\langle H_{(\varepsilon_m+\varepsilon_n)/2}, H_{(\varepsilon_m-\varepsilon_n)/2}\right\rangle = \frac{1}{4}\left(\frac{1}{\omega(x_m)} - \frac{1}{\omega(x_n)}\right) \quad \text{for } m < n, \tag{11}$$

$$\left\langle H_{(\varepsilon_m+\varepsilon_n)/2}, H_{(\varepsilon_m+\varepsilon_n)/2}\right\rangle = \frac{1}{4}\left(\frac{1}{\omega(x_m)} + \frac{1}{\omega(x_n)}\right) \quad \text{for } m < n, \tag{12}$$

so if $\alpha = (\varepsilon_m + \varepsilon_n)/2$, $\beta = (\varepsilon_m - \varepsilon_n)/2$, we get $|\langle H_\alpha, H_\beta\rangle| \leqslant \langle H_\alpha, H_\alpha\rangle$.

LEMMA 2. *The differential equation*

$$0 = y'' + \left(\operatorname{sech}^2(b+t) - \tanh^2(b+t)\right)y, \tag{13}$$

has the general solution

$$y = c_1\sinh(b+t) + (c_1 t + c_2)\operatorname{sech}(b+t), \tag{14}$$

and the solution satisfying $y(0) = 0$, $y'(0) = 1$ *is given by*

$$y = f(t) = \tfrac{1}{2}\{\operatorname{sech} b\sinh(b+t) + (t\operatorname{sech} b - \sinh b)\operatorname{sech}(b+t)\}. \tag{14$'$}$$

Further, for any $b < -\operatorname{arcsinh}(6\sqrt{2})/2$, *if* t_0 *is defined by the conditions* $b + t_0 > 0$, $\cosh^2(b+t_0) = 2$, *then* $f'(t_0) < 0$.

PROOF. That (14)′ gives the solution is just direct computation. For the rest, note that $b < -\operatorname{arcsinh}(6\sqrt{2})/2$ implies $2\sinh b\cosh b = \sinh(2b) < -6\sqrt{2}$ which implies $3\sqrt{2}\operatorname{sech} b + \sinh b < 0$. By definition, we have $b < 0$, $t_0 > 0$, $\cosh(b+t_0) = \sqrt{2}$, $\sinh(b+t_0) = 1$. Direct computation gives

$$f'(t_0) = \tfrac{1}{4}\left(3\sqrt{2}\operatorname{sech} b - t_0\operatorname{sech} b + \sinh b\right).$$

Since $t_0\operatorname{sech} b > 0$, we get $f'(t_0) < 0$.

THEOREM 2. *There exist bounded homogeneous domains D which in the Bergman metric have focal points* (*in the sense of* [**9**, **13**]).

PROOF. Fix $b < -\operatorname{arcsinh}(6\sqrt{2})/2$ and t_0 as in the Lemma. Consider the differential equation

$$0 = y'' + \left\{u\operatorname{sech}^2(b+t) - u^2\tanh^2(b+t)\right\}y, \tag{10$'$}$$

where u is a parameter, $0 < u \leqslant 1$. Let f_u be the solution satisfying the initial condition $f_u(0) = 0$, $f_u'(0) = 1$. We may rewrite (10)′ (and (13)) as a system of two first order equations involving the unknowns y, y'. Apply [**2**, p. 58] to that system for t in some compact interval containing 0 and t_0 in the interior. We conclude that there is a $\delta > 0$ such that $f_u'(t_0) < 0$ for $1 - \delta < u \leqslant 1$.

Recall that for the Bergman metric, one has [**3**],

$$\omega(X_k) = c\left(1 + \frac{1}{4}\dim \mathfrak{n}_{\varepsilon_k/2} + \frac{1}{2}\sum_{m\neq k}\dim \mathfrak{n}_{(\varepsilon_k+\varepsilon_m)/2}\right), \tag{15}$$

where $c > 0$ is independent of k. We claim (see following construction), that there exists a sequence of normal j-algebras so that, for a fixed pair of indices $m < n$, we have always $\dim \mathfrak{n}_{(\varepsilon_m-\varepsilon_n)/2} \geqslant 2$ and so that the dimension of $\mathfrak{n}_{\varepsilon_m/2}$ grows arbitrarily large while the dimensions of all other root spaces (and $\mathfrak{a}$) stay constant. By (15), one can make $\omega(X_m) - \omega(X_n)$ positive and $\omega(X_n)/\omega(X_m)$ arbitrarily close to 0. Let $\alpha = (\varepsilon_m + \varepsilon_n)/2$, $\beta = (\varepsilon_m - \varepsilon_n)/2$. By (11), we eventually have normal j-algebras

with $\langle H_\alpha, H_\beta \rangle < 0$ and by (12), we eventually get normal j-algebras with

$$\frac{\langle H_\alpha, H_\beta \rangle}{\langle H_\alpha, H_\alpha \rangle} = \frac{\omega(X_n)/\omega(X_m) - 1}{\omega(X_n)/\omega(X_m) + 1}$$

within δ of -1. Fixing such an algebra, we now choose c so as to obtain the normalization $|H_\alpha| = 1$.

Now since $\dim \mathfrak{n}_\alpha \geqslant 2$, we can pick unit vectors $Y_\alpha \in \mathfrak{n}_\alpha$, $Z_\beta \in \mathfrak{n}_\beta$ so that $\langle Y_\alpha, jZ_\beta \rangle = 0$. One knows that $[Y_\alpha, Z_\beta] \in \mathfrak{n}_{\alpha+\beta} = \mathfrak{n}_{\varepsilon_m}$ so $[Y_\alpha, Z_\beta] = rX_m$. Then $0 = \langle Y_\alpha, jZ_\beta \rangle = \omega[Y_\alpha, Z_\beta] = r\omega(X_m)$ shows that $r = 0$ and $[Y_\alpha, Z_\beta] = 0$. Since Y_α, jZ_β are in the same root space $\mathfrak{n}_\alpha$ and 2α is not a root, one sees from (2) that $\nabla_{Y_\alpha} jZ_\beta \in \mathfrak{a}$ and in fact

$$j\nabla_{Y_\alpha} Z_\beta = \nabla_{Y_\alpha} jZ_\beta = \langle Y_\alpha, jZ_\beta \rangle H_\alpha = 0.$$

Thus $\nabla_{Y_\alpha} Z_\beta = \nabla_{Z_\beta} Y_\alpha = 0$.

Let γ be a geodesic with $\gamma'(t) = -\tanh(b+t)H_\alpha + \operatorname{sech}(b+t)Y_\alpha$. Let $u = -\langle H_\alpha, H_\beta \rangle$. Comparing (10) and (10)′, we see that Proposition 2 says that $f_u Z_\beta$ is a nontrivial Jacobi field along γ vanishing at $\gamma(0)$, and by the construction of u and t_0, we see that $(d/dt)|f_u Z_\beta|^2$ is nonpositive at t_0. By [**9**, **13**], this means D has a focal point.

Construction. We indicate how to construct the sequence of normal j-algebras used in the previous proof. Start with any normal j-algebra $(\mathfrak{s}, j)$ with admissible form ω. Choose a 2-dimensional real vector space $\mathfrak{V}$ with a fixed basis $\{e, f\}$. Let $\mathfrak{s}' = \mathfrak{s} \oplus \mathfrak{V}$ (as vector spaces) and extend j to $\mathfrak{s}'$ by setting $j(e) = f$, $j(f) = -e$. Extend the bracket product to $\mathfrak{s}'$ so that $\operatorname{ad} jX_1 | \mathfrak{V} = I/2$, $[\Sigma_{k \neq 2} jX_k \oplus \mathfrak{n}, \mathfrak{V}] = 0$ and $[e, f] = -X_1/\omega(X_1)$. Extend ω to $\mathfrak{s}'$ so that $\omega(\mathfrak{V}) = 0$. A tedious but straightforward check shows that $(\mathfrak{s}', j)$ is a normal j-algebra with admissible form ω. Further, $\mathfrak{n}' = [\mathfrak{s}', \mathfrak{s}'] = \mathfrak{n} \oplus \mathfrak{V}$ so $\mathfrak{a}' = (\mathfrak{n}')^\perp = \mathfrak{a}$. The root spaces of the adjoint action of $\mathfrak{a}$ on $\mathfrak{s}'$ are just

$$\mathfrak{n}'_\alpha = \mathfrak{n}_\alpha \quad \text{for } \alpha \neq \tfrac{1}{2}\varepsilon_1, \qquad \mathfrak{n}'_{\varepsilon_1/2} = \mathfrak{n}_{\varepsilon_1/2} \oplus \mathfrak{V}.$$

It is easy to find normal j-algebras with $\dim \mathfrak{n}_{(\varepsilon_1 - \varepsilon_2)/2} \geqslant 2$. Pick such a one and iterate the above construction to find the desired sequence.

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