FACTORIZATION IN CODIMENSION TWO IDEALS
OF GROUP ALGEBRAS

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Abstract. Let $G$ be a finitely generated group and $I$ be a closed, two-sided ideal
with codimension two in $L^1(G)$. Then the linear span of the set of all products in $I$ is
equal to $I$.

Let $A$ be a complex algebra. For $I, J \subseteq A$, define $IJ = \{ \sum_{k=1}^n a_k b_k \mid n \in \mathbb{N};
\forall k \leq n, a_k \in I, b_k \in J \}$, and abbreviate $II$ as $I^2$. It is clear that if $I$ is an ideal
in $A$, then $I^2 \subseteq I$ (by ideal will always be meant a two-sided ideal). The ideal $I$ is
said to be idempotent if $I^2 = I$.

Now suppose that $A$ is also a Banach algebra. Then a question which arises in
connection with automatic continuity problems for $A$ is whether $I$ is idempotent
whenever $I$ is a closed finite-codimensional ideal in $A$. For example, see [2, §6] and
[3].

This question is particularly interesting when $A$ is the group algebra, $L^1(G)$, of
a locally compact group $G$, as there are several classes of groups such that every
finite-codimensional ideal in $L^1(G)$ is idempotent. For example, it is an immediate
consequence of Theorem 2 in [5] that, if $G$ is amenable, then every closed,
finitely-codimensional ideal in $L^1(G)$ has a bounded approximate identity. Hence, by
Cohen’s factorization theorem [1, Theorem 11.10], every such ideal is idempotent. It
is also shown in [8] that, if $G$ is connected, then every closed, finitely-codimensional
ideal in $L^1(G)$ is idempotent. Furthermore, in [6] it is shown that closed ideals with
codimension one in $L^1(G)$ are idempotent for every $G$. Every group algebra has
at least one codimension one ideal, namely the augmentation ideal $I_0(G) = \{ f \in L^1(G) \mid \int_G f \, dx = 0 \}$.

This paper is concerned with ideals with codimension two in $L^1(G)$. The main
theorem deals with finitely generated groups. Since a finitely generated group $G$ is
countable, Haar measure on $G$ will be discrete. Hence we may normalize it to be
counting measure and take $L^1(G)$ to be the set of functions $f$ on $G$ with

$$
\|f\| = \sum_{x \in G} |f(x)| < \infty.
$$

The function which takes the value one at $x$ and is zero elsewhere will be denoted by
$x$. If $1$ is the identity element of $G$, then $\bar{1}$ is a multiplicative identity for $L^1(G)$.

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Theorem 1. Let $G$ be a finitely generated group and $I$ be a closed, two-sided ideal with codimension two in $L^1(G)$. Then $I$ is idempotent.

Proof. Consider $L^1(G)/I$. It is a two-dimensional complex algebra with unit, and is semisimple by [3, Lemma 3.1]. (Alternatively, $L^1(G)/I$ may be shown to be semisimple in the following way. Let $\rho : L^1(G) \to L^1(G)/I$ be the quotient homomorphism and $R$ be the radical of $L^1(G)/I$. Then if $R \neq (0)$, $\rho^{-1}(R)$ is a non-idempotent codimension one ideal in $L^1(G)$, which contradicts the theorem of [6].) Hence $L^1(G)/I$ is isomorphic to $\mathbb{C} \oplus \mathbb{C}$ and so there are multiplicative linear functionals $\phi_0$ and $\phi_1$ on $L^1(G)$ such that $I = \ker \phi_0 \cap \ker \phi_1$.

Let $\chi_0$ and $\chi_1$ be the characters on $G$ such that
\[
\phi_j(f) = \sum_{x \in G} f(x)\chi_j(x) \quad (f \in L^1(G), j = 0, 1)
\]
(see [4, Corollary 23.7]). Define an automorphism $T$ of $L^1(G)$ by
\[
(Tf)(x) = f(x)\chi_0(x) \quad (x \in G, f \in L^1(G)).
\]
Then $T(I)$ is a codimension two ideal in $L^1(G)$ and in order to show that $I$ is idempotent it will suffice to show that $T(I)$ is idempotent. Hence, replacing $I$ by $T(I)$ and $\phi_0 \circ T^{-1}$, we may suppose that $\chi_0$ is the trivial character and $\ker \phi_0$ is the augmentation ideal of $L^1(G)$.

Now suppose that $G$ is generated by $n$ elements $y_1, y_2, \ldots, y_n$ and let $F_n$ be the free group on $n$ generators $x_1, x_2, \ldots, x_n$. Then there is a surjective group homomorphism $\varphi : F_n \to G$ defined by
\[
\varphi(x_i) = y_i, \quad i = 1, 2, \ldots, n,
\]
and a surjective algebra homomorphism $Q : L^1(F_n) \to L^1(G)$ defined by
\[
(Qf)(y) = \sum_{x \in \varphi^{-1}(y)} f(x) \quad (y \in G, f \in L^1(F_n)).
\]
It is clear that $Q^{-1}(I)$ is a codimension two ideal in $L^1(F_n)$ and that $I$ is idempotent if $Q^{-1}(I)$ is. Hence it will suffice to prove the theorem in the case when $G = F_n$.

If $\chi_i$ was the trivial character, then we would have that $\phi_0 = \phi_i$ and $I$ would have codimension one. Thus $\chi_i$ is not trivial and we may suppose that $\chi_i(x_i) \neq 1$. Now if $\chi_i(x_i) = 1$ for some $i$, then $\chi_i(x_1x_i) \neq 1$ and $\{x_1, \ldots, x_{i-1}, x_1x_i, x_{i+1}, \ldots, x_n\}$ still generates $F_n$ freely. Thus, by replacing $x_i$ with $x_ix_i$ if necessary, we may further suppose that $\chi_i(x_i) \neq 1$ for each $i$.

It is convenient to introduce a little more notation. Let $2^n$ denote the set of all functions on $\{1, 2, \ldots, n\}$ taking values 0 or 1. For each $t$ in $2^n$, define $e(t)$ in $L^1(F_n)$ by $e(t) = (c_1 \bar{1} - \bar{x}_i) \ast (c_2 \bar{1} - \bar{x}_2) \ast \cdots \ast (c_n \bar{1} - \bar{x}_n)$, where $c_i = 1$ if $t(i) = 0$ and $c_i = \chi_i(x_i)$ if $t(i) = 1$. The constant functions in $2^n$ with values 0 and 1 will be denoted by 0 and 1 respectively. We will require the following lemma.

Lemma. Let $t$ be in $2^n \setminus \{0, 1\}$. Then $e(t)$ is in $I^2$.

Proof. Since $t$ is not constant, there is an $i$ between 1 and $n - 1$ such that either
\[
e(t) = \cdots \ast (\bar{1} - \bar{x_i}) \ast (\chi_1(x_{i+1})\bar{1} - \bar{x}_{i+1}) \ast \cdots
\]
or
\[ e(t) = \cdots \ast (\chi_i(x_i) \tilde{1} - \bar{x}_i) \ast (\tilde{1} - \bar{x}_{i+1}) \ast \cdots. \]
Thus, since \( I^2 \) is an ideal in \( L^1(\mathbb{F}_n) \), it will suffice to show that
\[
(\tilde{1} - \bar{x}_i) \ast (\chi_i(x_{i+1}) \tilde{1} - \bar{x}_{i+1}) \quad \text{and} \quad (\chi_i(x_i) \tilde{1} - \bar{x}_i) \ast (\tilde{1} - \bar{x}_{i+1})
\]
are in \( I^2 \) for each \( i \) between 1 and \( n - 1 \).

For this, choose \( h \) in \( L^1(\mathbb{F}_n) \) such that \( h \) is in \( \ker \phi_1 \) and \( \tilde{1} - h \) is in \( \ker \phi_0 \). Then
\[
\begin{align*}
(1) \quad (\tilde{1} - \bar{x}_i) \ast (\chi_i(x_{i+1}) \tilde{1} - \bar{x}_{i+1}) &= h \ast (\tilde{1} - \bar{x}_i) \ast (\chi_i(x_{i+1}) \tilde{1} - \bar{x}_{i+1}) \\
+ (\tilde{1} - h) \ast (\chi_i(x_{i+1}) \tilde{1} - \bar{x}_{i+1}) \ast (\tilde{1} - \bar{x}_i) + (\tilde{1} - h) \ast (\chi_i(x_{i+1} - \bar{x}_{i+1})x_i).
\end{align*}
\]
Now \( \tilde{1} - \bar{x}_i \) has a square root defined by the binomial expansion
\[
(\tilde{1} - \bar{x}_i)^{1/2} = \sum_{k=0}^{\infty} \binom{1/2}{k} (\bar{x}_i)^k,
\]
where the series converges because \( (1/2)^k \) is an \( l^1 \)-sequence and because \( ||(\bar{x}_i)^k|| = 1 \) for each \( k \). The square root is in \( \ker \phi_0 \) because \( 0 = \phi_0(\tilde{1} - \bar{x}_i) = \phi_0((1 - \bar{x}_i)x_i^{1/2})^2 \).

Hence
\[
h \ast (\tilde{1} - \bar{x}_i) \ast (\chi_i(x_{i+1}) \tilde{1} - \bar{x}_{i+1})
\]
\[
= \left[ h \ast (\tilde{1} - \bar{x}_i)^{1/2} \right] \ast \left[ (\tilde{1} - \bar{x}_i)^{1/2} \ast (\chi_i(x_{i+1}) \tilde{1} - \bar{x}_{i+1}) \right]
\]
\[
\in \left[ \ker \phi_1 \ast \ker \phi_0 \right] \ast \left[ \ker \phi_0 \ast \ker \phi_1 \right] \subseteq I^2.
\]

It may be shown in the same way that \( (\tilde{1} - h) \ast (\chi_i(x_{i+1}) \tilde{1} - \bar{x}_{i+1}) \ast (\tilde{1} - \bar{x}_i) \) is in \( I^2 \). Finally,
\[
\chi_i(x_{i+1} - \bar{x}_{i+1}x_i) = (\tilde{1} - (x_{i+1}x_i^{1}x_{i+1}^{1})^{-1}) \ast (\bar{x}_i^{1}x_{i+1}^{1}),
\]
where \( \tilde{1} - (x_{i+1}x_i^{1}x_{i+1}^{1})^{-1} \) is in \( I \) and has a square root which must also be in \( I \)
because \( I = \ker \phi_0 \cap \ker \phi_1 \). Hence \( x_{i+1}x_i^{1} - x_i^{1}x_{i+1} \) is in \( I^2 \) also. It now follows from (1) that \( (\tilde{1} - \bar{x}_i) \ast (\chi_i(x_{i+1}) \tilde{1} - \bar{x}_{i+1}) \) is in \( I^2 \) for each \( i \). That \( (\chi_i(x_i) \tilde{1} - \bar{x}_i) \ast (\tilde{1} - \bar{x}_{i+1}) \) is in \( I^2 \) also may be proved in the same way.

We now continue the proof of the theorem. Every \( x \) in \( \mathbb{F}_n \) may be written uniquely in the form \( x = x_1^1x_2^2 \cdots x_n^nx_0y \), where \( k_1, k_2, \ldots, k_n \) are integers and \( y \) is in \( \mathbb{F}_n' \), the commutator subgroup of \( \mathbb{F}_n \). For each \( i \) between 1 and \( n \), let \( Z_i = \langle x_i^k \cdots x_n^ny | k_1, \ldots, k_n \in \mathbb{Z}, y \in \mathbb{F}_n' \rangle \) and let \( \langle x_i \rangle \) be the subgroup of \( \mathbb{F}_n \) generated by \( x_i \). Let \( Z_{n+1} = \mathbb{F}_n' \). Then \( Z_i = \mathbb{F}_n \) and \( Z_{i+1} \subseteq Z_i \), \( i = 1, 2, \ldots, n \).

Let \( i \) be between 1 and \( n \) and denote by \( L^1(\langle x_i \rangle) \) and \( L^1(Z_i) \) the spaces of functions in \( L^1(\mathbb{F}_n) \) with support in \( \langle x_i \rangle \) and \( Z_i \) respectively. Since \( \chi_i(x_i) \neq 1 \), \( L^1(\langle x_i \rangle) \cap I \) has codimension two in \( L^1(\langle x_i \rangle) \). Hence we may define bounded linear functionals \( \eta, \xi \) on \( L^1(\langle x_i \rangle) \) and a bounded linear operator \( T: L^1(\langle x_i \rangle) \to L^1(\langle x_i \rangle) \cap I \) in a unique way such that for each \( f \) in \( L^1(\langle x_i \rangle) \),
\[
(2) \quad f = \eta(f)(\tilde{1} - \bar{x}_i) + \xi(f)(\chi_i(x_i) \tilde{1} - \bar{x}_i) + T(f).
\]
Furthermore, $L^1(\langle x, \rangle)$ is a subalgebra of $L^1(F_n)$ isomorphic to $L^1(\mathbb{Z})$ (the group algebra of the integers) and $L^1(\langle x, \rangle) \cap I$ is a closed ideal in $L^1(\langle x, \rangle)$. Hence, by [5, Theorem 2], $L^1(\langle x, \rangle) \cap I$ has a bounded approximate identity. Let $J_i$ be the closed linear span of $(L^1(\langle x, \rangle) \cap I) \ast L^1(F_n)$. Then it is clear that $J_i$ is a left Banach module over $L^1(\langle x, \rangle) \cap I$, that $J_i$ is contained in $I$, and that $L^1(\langle x, \rangle) \cap I$ has a bounded approximate identity for $J_i$ (in the sense of [1, 11.8]).

Now for each $f$ in $L^1(Z_n)$ and each $z$ in $Z_{n+1}$, let $f_z$ be the function in $L^1(\langle x, \rangle)$ defined by $f_z(x^k) = f(x^k z)$. Then $\Sigma_{z \in Z_{n+1}} \|f_z\| = \|f\|$ and $f = \Sigma_{z \in Z_{n+1}} f_z \ast \bar{z}$. Hence, by (2),

$$f = (\bar{x} - \bar{x}_i) \ast \left( \Sigma_{z \in Z_{n+1}} \eta(f_z) \bar{z} \right) + (\chi_i(x_i) \bar{z} - \bar{x}_i) \ast \left( \Sigma_{z \in Z_{n+1}} \xi(f_z) \bar{z} \right)$$

Put $h = \Sigma_{z \in Z_{n+1}} T(f_z) \ast \bar{z}$. Then $h$ is in $J_i$ and so, by Cohen’s factorization theorem [1, 11.10], there are $a$ in $L^1(\langle x, \rangle) \cap I$ and $h'$ in $J_i$ such that $h = a \ast h'$. Since both $L^1(\langle x, \rangle) \cap I$ and $J_i$ are contained in $I$, it follows that $h$ is in $I^2$. Hence, putting $\Sigma_{z \in Z_{n+1}} \eta(f_z) \bar{z} = g_0$ and $\Sigma_{z \in Z_{n+1}} \xi(f_z) \bar{z} = g_1$, we have shown that for each $f$ in $L^1(Z_n)$, (3)

$$f = (\bar{z} - \bar{x}_i) \ast g_0 + (\chi_i(x_i) \bar{z} - \bar{x}_i) \ast g_1 + h,$$

where $g_0$ and $g_1$ are in $L^1(Z_{n+1})$ and $h$ is in $I^2$.

In particular, if $f$ is in $L^1(F_n)$, then

$$f = (\bar{z} - \bar{x}_i) \ast g_0 + (\chi_i(x_i) \bar{z} - \bar{x}_i) \ast g_1 + h,$$

where $g_0$ and $g_1$ are in $L^1(Z_n)$ and $h$ is in $I^2$. Next, applying (3) to $g_0$ and $g_1$ and remembering that $I^2$ is an ideal, we find that

$$f = (\bar{z} - \bar{x}_i) \ast (\bar{z} - \bar{x}_2) \ast g_{00} + (\bar{z} - \bar{x}_i) \ast (\chi_i(x_2) \bar{z} - \bar{x}_2) \ast g_{01} + (\chi_i(x_1) \bar{z} - \bar{x}_1) \ast (\bar{z} - \bar{x}_2) \ast g_{10} + (\chi_i(x_1) \bar{z} - \bar{x}_1) \ast (\chi_i(x_2) \bar{z} - \bar{x}_2) \ast g_{11} + h''$$

where $g_{00}$, $g_{01}$, $g_{10}$, $g_{11}$ are in $L^1(Z_3)$ and $h''$ is in $I^2$. Now applying (3) to $g_{00}$, $g_{01}$, $g_{10}$ and $g_{11}$ and so on, we find after $n$ steps that for each $f$ in $L^1(F_n)$,

$$f = \sum_{t \in \mathbb{Z}_n} e(t) \ast g(t) + \psi,$$

where $g(t)$ is in $L^1(Z_{n+1}) (= L^1(F_n))$ for every $t$ in $2^n$ and $\psi$ is in $I^2$. Finally, by the lemma, $e(t)$ is in $I^2$ for each $t$ other than $0$ or $I$ and so for every $f$ in $L^1(F_n)$,

$$f = e(0) \ast g(0) + e(1) \ast g(1) + \psi',$$

where $g(0)$ and $g(1)$ are in $L^1(F_n')$ and $\psi'$ is in $I^2$.

Let $f$ be in $I$. Then,

$$0 = \phi_1(f) = \phi_1(e(0))\phi_1(g(0)) + \phi_1(e(1))\phi_1(g(1)) + \phi_1(\psi').$$

Since $\phi_1(\psi') = 0 = \phi_1(e(1))$ and $\phi_1(e(0)) \neq 0$, it follows that $\phi_1(g(0)) = 0$. Thus,

$$0 = \sum_{x \in F_n} g(0)(x)\chi_1(x) = \sum_{y \in F_n} g(0)(y).$$
because $g(0)$ is supported in $I_0(F'_n)$ and characters are trivial on $F'_n$. Hence $g(0)$ is in $I_0(F'_n)$. Similarly, using the fact that $\phi_0(f) = 0$, it may be shown that $g(I)$ is in $I_0(F'_n)$. Now $I_0(F'_n)$ is contained in $I$ and is idempotent by [6]. Hence,

$$g(0), g(I) \in I_0(F'_n) = I_0(F'_n)^2 \subseteq I^2,$$

and so $f$ is in $I^2$. Therefore $I$ is idempotent.

There are still many unanswered questions concerning factorization in finite-codimensional ideals of group algebras. Some of these questions are discussed in more detail in \S4 of [7]. It is mentioned there, without proof, that the above theorem may be used to show that, if $G$ is finitely generated, then every ideal $I$ such that $L^1(G)/I$ is commutative and finite dimensional is idempotent. In particular it follows that, if $G$ is finitely generated, then every ideal with codimension three in $L^1(G)$ is idempotent. We now give a proof of this fact.

**Theorem 2.** Let $A$ be a complex algebra such that $A^2 = A$. Suppose that

(i) every ideal with codimension one in $A$ is idempotent; and  
(ii) if $I$ and $J$ are ideals with codimension one in $A$, then $II = JI$.

Then every ideal $I$ in $A$ such that $A/I$ is finite dimensional and commutative is idempotent.

**Proof.** By adjoining an identity we may assume that $A$ has a unit, which we will denote by $1$. Let $I$ be an ideal in $A$ with $A/I$ commutative and finite dimensional, but with $\dim(A/I) > 2$. We may suppose that every ideal properly containing $I$ is idempotent.

Denote by $\rho$ the quotient map from $A$ to $A/I$ and let $R$ be the radical of $A/I$. If $R \neq (0)$, then $R^2 \subseteq R$ because $R$ is a finite-dimensional radical algebra. Hence, if $R \neq (0)$, $(\rho^{-1}(R))^2 \subseteq \rho^{-1}(R^2) = \rho^{-1}(R)$, and so $\rho^{-1}(R)$ is a nonidempotent ideal in $A$ which properly contains $I$. Therefore $R = (0)$, and so $A/I$ is semisimple. It follows that $A/I$ is isomorphic to the direct sum of $n$ copies of $C$, where $n$ is the codimension of $I$. Hence, there are ideals $I_1, I_2, \ldots, I_n$ with codimension one in $A$ such that $I = \bigcap_{j=1}^n I_j$.

Since $I_1$ and $I_2$ are distinct codimension one ideals in $A$, there is an $h$ in $I_1$ such that $1 - h$ is in $I_2$. Hence, for each $f$ in $I$, $f = fh + f(1 - h)$, and so $I \subseteq II_1 + II_2$.

Similarly, if $j_1, j_2, \ldots, j_k$ are distinct integers between 1 and $n$, where $k \leq n - 2$, and $i_1$ and $i_2$ are distinct integers not equal to any of the $j_k$'s then

$$II_{j_1}I_{j_2} \cdots I_{j_k} \subseteq II_{j_1} \cdots I_{j_k}I_{i_1} + II_{j_1} \cdots I_{j_k}I_{i_2},$$

where now $\{j_1, j_2, \ldots, j_k, i_1\}$ and $\{j_1, j_2, \ldots, j_k, i_2\}$ are sets of distinct integers. We may thus show that $I$ is contained in the sum of ideals of the form $II_{j_1}I_{j_2} \cdots I_{j_n}$, where $j_1, j_2, \ldots, j_n$ are distinct integers between 1 and $n$. However, $I \subseteq \bigcap_{j=1}^n I_j$, and so $II_{j_1}I_{j_2} \cdots I_{j_n} \subseteq I_{j_1}I_{j_2} \cdots I_{j_n}$, where $j_1$ is chosen to be the integer between 1 and $n$ which does not appear in the list $j_2, j_3, \ldots, j_n$. Therefore,

$$I \subseteq \sum_{\pi \in S_n} (I_{\pi(1)}I_{\pi(2)} \cdots I_{\pi(n)}),$$

where $S_n$ is the group of permutations of $\{1, 2, \ldots, n\}$. 


Now,
\[ I_1 I_2 \cdots I_n = (I_1)^2 (I_2)^2 \cdots (I_n)^2, \text{ by (i)}, \]
\[ = I_1 I_2 I_1 I_2 (I_3)^2 \cdots (I_n)^2, \text{ by (ii)}, \]
\[ = (I_1 I_2 I_3)^2 (I_4)^2 \cdots (I_n)^2, \text{ by (ii) applied twice}, \]
\[ = \cdots \]
\[ = (I_1 I_2 \cdots I_n)^2 \subseteq I^2. \]

Similarly, \( I_{\pi(1)} I_{\pi(2)} \cdots I_{\pi(n)} \subseteq I^2 \) for every \( \pi \) in \( S_n \). Therefore, by (4), \( I \subseteq I^2 \).

Remarks. (a) The codimension one ideals of \( A \) will satisfy (ii) either if \( A \) is commutative (clear) or if the codimension two ideals of \( A \) are idempotent \( (IJ \subseteq I \cap J = (I \cap J)^2 \subseteq JJ) \).

(b) If \( A \) is a Banach algebra and \( A^2 = A \), then every codimension one ideal in \( A \) is closed. Hence we need only verify that (i) and (ii) hold for the closed codimension one ideals of \( A \). Furthermore, in the course of the proof it was shown that if \( A \) satisfies (i) and (ii) and if \( I \) is an ideal in \( A \) such that \( A/I \) is commutative and finite dimensional, then \( I \) is the intersection of codimension one ideals. Hence under these conditions \( I \) is closed.

(c) Suppose that \( A \) satisfies the conditions of the theorem, then all ideals with codimension two or three in \( A \) are idempotent. If \( I \) has codimension two, then \( A/I \) is semisimple (otherwise the inverse image of \( \text{Rad}(A/I) \) will be a nonidempotent codimension one ideal). Hence \( A/I \) is isomorphic to \( \mathbb{C} \oplus \mathbb{C} \) (the only semisimple, two-dimensional complex algebra). Now \( \mathbb{C} \oplus \mathbb{C} \) is commutative and so \( I \) is idempotent by the theorem. A similar argument proves the result for codimension three ideals. The result does not necessarily hold for codimension four ideals because there is a four-dimensional noncommutative, semisimple complex algebra—namely the \( 2 \times 2 \) complex matrices.

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