A NOTE ON NEW GENERATING RELATIONS  
FOR FUNCTIONS OF SEVERAL VARIABLES

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Abstract. In this paper we have derived generating relations for functions of several variables by using the operator $T_k = x(k + xD)$ and the operational relations involving this operator. Some recent results of Srivastava and Panda [7] have been conveniently obtained by this method as well as some hitherto unknown results established.

Introduction. Lauricella's function of several variables is defined as follows

\begin{equation}
F_D^{(r)}[\{\alpha; m_i\}; \{\beta_i; \phi_i\}; \{\gamma; \Psi_i\}; x_1, \ldots, x_r] = \sum_{k_1, \ldots, k_r = 0}^{\infty} \frac{(\alpha)_{\sum_k k_i} (\beta_1)_{k_1} \cdots (\beta_r)_{k_r}}{\gamma_{\sum_k k_i} \Psi_i} \frac{x_1^{k_1} \cdots x_r^{k_r}}{k_1! \cdots k_r!},
\end{equation}

where the parameters $\alpha, \beta, \gamma$ are arbitrary complex numbers, the coefficients $m_i, \phi_i, \Psi_i$ are real and positive and for the convergence of multiple series

\begin{equation}
1 + \sum_{i=1}^{r} \phi_i \Psi_i - m_i \phi_i \geq 0, \quad i = 1, 2, \ldots, r,
\end{equation}

the equality holds only if $|x_1|, |x_2|, \ldots, |x_r|$ are constrained appropriately. In particular if $m_i = m, \Psi_i = \Psi, i = 1, 2, \ldots, r$, then the equality in (1.2) will hold when either

\begin{equation}
m > \Psi \quad \text{and} \quad \sum_{i=1}^{r} \left( \frac{|x_i|}{G_i} \right)^{1/(m - \Psi)} < 1
\end{equation}

or

\begin{equation}
m \leq \Psi \quad \text{and} \quad \max_{1 \leq i \leq r} \left( \frac{|x_i|}{G_i} \right) < 1,
\end{equation}

where

\begin{equation}
G_i = m^{-m} \phi_i \Psi_i, \quad i = 1, 2, \ldots, r.
\end{equation}
In this paper we have found new generating relations for Lauricella’s function and some related function of several variables, viz.

\[(1.6) \sum_{n_1,\ldots,n_r=0}^{\infty} \frac{(\lambda_1)_2 n_1 \cdots (\lambda_r)_2 n_r, t_1^{n_1} \cdots t_r^{n_r}}{(\lambda_1)_n \cdots (\lambda_r)_n, n_1! \cdots n_r!} F_{D}^{(r)}[(\lambda; m_i): (-n; m_i); (\gamma; \Psi_i); x_1,\ldots,x_r] = \prod_{i=1}^{r} \left\{ \frac{(1 - 4 t_i)^{1/2} \left[ \frac{2}{1 + (1 - 4 t_i)^{1/2}} \right]}{(1 - 4 t_i)^{1/2} \left[ \frac{2}{1 + (1 - 4 t_i)^{1/2}} \right]} \right\} \cdot F_{D}^{(r)}[(\lambda; m_i): (1 - \lambda_i - 2 n_i; m_i); (\gamma; \Psi_i); x_1 \left( \frac{2 t_1}{1 + (1 - 4 t_1)^{1/2}} \right)^{m_1}, \ldots, x_r \left( \frac{2 t_r}{1 + (1 - 4 t_r)^{1/2}} \right)^{m_r}].\]

Recently Srivastava and Panda \[7\] considered the functions

\[(1.7) \sum_{k_1,\ldots,k_r=0}^{\infty} C(k_1,\ldots,k_r) \frac{z_1^{k_1}}{k_1!} \cdots \frac{z_r^{k_r}}{k_r!},\]

where the coefficients \(C(k_1,\ldots,k_r), k_i \geq 0, i = 1,2,\ldots,r\), are arbitrary constants real or complex.

\[(1.8) \Delta_{n_1,\ldots,n_r; q_1,\ldots,q_r}(z_1,\ldots,z_r) = \sum_{k_1=0}^{[n_1/q_1]} \cdots \sum_{k_r=0}^{[n_r/q_r]} C(k_1,\ldots,k_r) \prod_{i=1}^{r} \left\{ \frac{(-n_i)_{q,k_i}}{(1 + \alpha_i + \beta_i n_i)_{q,k_i, k_i!}} z_i^{k_i} \right\}\]

where \(\alpha_i\) and \(\beta_i\) are parameters independent of \(n_1,\ldots,n_r\) and \(q_i\), are arbitrary positive integers, \(i = 1,\ldots,r\) and

\[(1.9) S_{n_1,\ldots,n_r; q_1,\ldots,q_r}^{(\alpha_1,\ldots,\alpha_r; \beta_1,\ldots,\beta_r)}(\lambda_1,\ldots,\lambda_r; z_1,\ldots,z_r) = \sum_{k_1=0}^{[n_1/q_1]} \cdots \sum_{k_r=0}^{[n_r/q_r]} C(k_1,\ldots,k_r) \prod_{i=1}^{r} \left\{ \frac{(-n_i)_{q,k_i}(1 + \alpha_i + (\beta_i + 1) n_i)_{\lambda,k_i}}{(1 + \alpha_i + \beta_i n_i)_{(\lambda + q_i)k_i, k_i!}} z_i^{k_i} \right\}\]

where \(\alpha_i, \beta_i\), and \(\lambda_i, i = 1,2,\ldots,r\), are complex parameters independent of \(n_1,\ldots,n_r\) and \(q_1,\ldots,q_r\) are arbitrary positive integers and derived the generating relation for them.

Besides these Srivastava \[5,6\], Sharma and Abiodun \[1\] and Carlitz and Srivastava \[10\] derived generating relations for (1.7) and the G function and obtained generating functions for Lauricella’s function of several variables. Shukla \[2\] using the operator \(T_k = x(k + xD)\) and the operational relations involving these operators derived the results of Srivastava \[6\], Sharma and Abiodun \[1\] and Carlitz and Srivastava \[10\] and established some hitherto unknown results.
In this paper we propose to derive the results of Srivastava and Panda [7] and other results by making use of operator formulas (2.1), (3.1), (3.4), (3.7) and (3.9).

In §2 we obtain generating relations for functions defined by (1.8) and (1.9) and in §3 we derive various generating functions for Lauricella’s function of several variables.

2. Generating relations for functions of several variables. In this section we shall use the Mittal [4] operational generating formula

$$\sum_{n=0}^{\infty} \frac{1}{n!} T_{a+1+(m-1)n} f(x) = \frac{(1 + \nu)^{a+1}}{1 - (m - 1)\nu} f(x(1 + \nu)),$$

where $\nu = x(1 + \nu)^m$, $m$ being constant and $f(x)$ admits a formal power series in $x$ and $T_k \equiv x(k + xD), D = d/dx$.

Assuming that the operator $T_{1,k}$ is the operator $T_k$ operating on $t_1$ alone and similarly $T_{r,k}$ the operator on $t_r$ alone and replacing $a$ by $a_1 = \alpha_1, \ldots, a_r = \alpha_r$, $m$ by $m_1 = \beta_1 + 1, \ldots, m_r = \beta_r + 1$, $n$ by $n_1, n_2, \ldots, n_r$ respectively and putting

$$f(x) \equiv f\left[ x_1(-\beta_1 t_1^{\beta_1+1})^{m_1}, \ldots, x_r(-\beta_r t_r^{\beta_r+1})^{m_r} \right]$$

in (2.1) $r$ times, we easily get

$$\sum_{n_1,\ldots,n_r=0}^{\infty} \left( \begin{array}{c} \alpha_1 + (\beta_1 + 1)n_1 \\ n_1 \end{array} \right) \cdots \left( \begin{array}{c} \alpha_r + (\beta_r + 1)n_r \\ n_r \end{array} \right) t_1^{n_1} t_2^{n_2} \cdots t_r^{n_r}$$

$$= \frac{(-n_i)_{m_{ki}}}{(1 + \alpha_i + \beta_i n_i)_{m_{ki}}} \frac{x_1^{k} y_{i-\beta_i, i+1} (1 + \nu_i)^{\beta_i+1} m_i}{k_i!}$$

$$= \frac{(1 + \nu_1)^{\alpha_1+1} \cdots (1 + \nu_r)^{\alpha_r+1}}{(1 - \beta_1 \nu_1) \cdots (1 - \beta_r \nu_r)} f\left[ x_1(-\beta_1 t_1^{\beta_1+1})^{m_1}, \ldots, x_r(-\beta_r t_r^{\beta_r+1})^{m_r} \right],$$

where $\nu_i = t_i(1 + \nu_i)^{\beta_i+1}, i = 1, 2, \ldots, r$, and $N = \min(n_1, n_2, \ldots, n_r)$. Now putting

$$y_i = t_i, i = 1, 2, \ldots, r, \text{ in (2.2) we get}$$

$$\sum_{n_1,\ldots,n_r=0}^{\infty} \left( \begin{array}{c} \alpha_1 + (\beta_1 + 1)n_1 \\ n_1 \end{array} \right) \cdots \left( \begin{array}{c} \alpha_r + (\beta_r + 1)n_r \\ n_r \end{array} \right)$$

$$= \frac{(1 + \nu_1)^{\alpha_1+1} \cdots (1 + \nu_r)^{\alpha_r+1}}{(1 - \beta_1 \nu_1) \cdots (1 - \beta_r \nu_r)} f\left[ x_1(-\nu_1)^{m_1}, \ldots, x_r(-\nu_r)^{m_r} \right],$$

where the function $\Delta$ is given by (1.8), which is due to Srivastava and Panda [7, Theorem 2]. Further with the similar assumptions on operator $T_k$, $a$ and $m$ as in the derivation of (2.3), but taking

$$f(x) \equiv f\left[ x_1(-1)^{m_1} y_1^{\beta_1, m_1-\lambda_1 t_1^{(1+\beta_1) m_1+\lambda_1}}, \ldots, x_r(-1)^{m_r} y_r^{\beta_r, m_r-\lambda_r t_r^{(1+\beta_r) m_r+\lambda_r}} \right]$$
in (2.1) and using it \( r \) times, we similarly have
\[
\sum_{n_1, \ldots, n_r=0}^{\infty} \alpha_1 + (\beta_1 + 1)n_1 \ldots \alpha_r + (\beta_r + 1)n_r
\]
\[
S^{(\alpha_1, \ldots, \alpha_r; \beta_1, \ldots, \beta_r)}[\lambda_1, \ldots, \lambda_r; x_1, \ldots, x_r] t_1^{n_1} \ldots t_r^{n_r}
\]
\[
= \prod_{i=1}^{r} \frac{(1 + v_i)^{a_i+1}}{(1 - \beta_i v_i)} f \left[ x_i (-v_i)^{m_i} (1 + \nu_i)^{\lambda_i}, \ldots, x_r (-v_r)^{m_r} (1 + \nu_r)^{\lambda_i} \right],
\]
where \( v_i = t_i (1 + \nu_i)^{\beta_i+1}, i = 1, 2, \ldots, r \), and the function \( S \) is given by (1.9). This is also due to Srivastava and Panda [7, Theorem 2.1].

3. Lauricella's function of several variables. We now make use of the Mittal [3] operational formula
\[
\sum_{n=0}^{\infty} \frac{1}{n!} T_n \{ f(x) \} = (1 - x)^{-a-1} f \left[ \frac{x}{1 - x} \right],
\]
where \( f(x) \) admits a formal power series in \( x \). Replacing \( a \) by \( a_1 = \lambda_1 - 1, \ldots, a_r = \lambda_r - 1, n \) by \( n_1, \ldots, n_r \) and assuming that \( T_{i,k} \) is the operator \( T_k \) operating on \( i \) alone for \( i = 1, 2, \ldots, r \) respectively and putting
\[
f(x) \equiv F_D^{(r)}[\lambda; m_1; (\beta_1; \phi_1); (\gamma; \Psi_1); x_1, \ldots, x_r (-t_j)^{m_j}]
\]
in (3.1) and using it \( r \) times, we get
\[
\sum_{n_1, \ldots, n_r=0}^{\infty} \frac{(\lambda_1)_{n_1} \ldots (\lambda_r)_{n_r} t_1^{n_1} \ldots t_r^{n_r}}{n_1! \ldots n_r!}
\]
\[
\prod_{k_1, \ldots, k_r=0}^{N} \frac{(\lambda)_{k_1, k_2} \ldots (\lambda)_{k_r, k_1} \ldots (\lambda)_{k_r, k_1}}{(\gamma)_{k_1, k_2} \ldots (\lambda)_{k_r, k_1} \ldots (\lambda)_{k_r, k_1}} x_1^{k_1} x_r^{k_r}
\]
\[
= \prod_{i=1}^{r} \left[ (1 - t_i)^{\lambda_i} \right] F_D^{(r)}[\lambda; m_i; (\beta_i; \phi_i); (\gamma; \Psi_i); x_1, \ldots, x_r]
\]
Taking \( \beta_r = \lambda_r \) and \( \phi_r = m_r \) in (3.2) we have the generating relation
\[
\sum_{n_1, \ldots, n_r=0}^{\infty} \frac{(\lambda_1)_{n_1} \ldots (\lambda_r)_{n_r} t_1^{n_1} \ldots t_r^{n_r}}{n_1! \ldots n_r!} F_D^{(r)}[\lambda; m_1; (\gamma; \Psi_1); x_1, \ldots, x_r]
\]
\[
= \prod_{i=1}^{r} \left[ (1 - t_i)^{\lambda_i} \right] F_D^{(r)}[\lambda; m_i; (\gamma; \Psi_i); x_1, \ldots, x_r]
\]
Again by making use of the Mittal [3] operational formula

\[
\sum_{n=0}^{\infty} \frac{1}{n!} T_a^n f(x) = (1 - 4x)^{-1/2} \left[ \frac{2}{1 + (1 - 4x)^{1/2}} \right]^{a-1} f \left[ \frac{2x}{1 + (1 - 4x)^{1/2}} \right],
\]

replacing \( a \) by \( a_i = \lambda_1, \ldots, a_r = \lambda_r, \) \( n \) by \( n_1, \ldots, n_r \) and assuming that \( T_{i,k} \) is the operator \( T_k \) operating on \( t_i \) alone, \( i = 1, 2, \ldots, r, \) respectively and taking

\[ f(x) \equiv F_D^{(r)}[(\lambda; m_1); (\beta_i; \phi_i); (\gamma; \Psi); x_1, t_1^{m_1}, \ldots, x_r, t_r^{m_r}] \]

in (3.4) and using it \( r \) times, we have the generating relation (3.5)

\[
\sum_{n_1, \ldots, n_r=0}^{\infty} \frac{(\lambda_1)_{2n_1} \cdots (\lambda_r)_{2n_r} t_1^{n_1} \cdots t_r^{n_r}}{(\lambda_1)_{n_1} \cdots (\lambda_r)_{n_r} n_1! \cdots n_r!} 
\cdot F_D^{(r)}[(\lambda; m_1); (\beta_i; \phi_i); (\gamma; \Psi); x_1, \left( \frac{2t_1}{1 + (1 - 4t_1)^{1/2}} \right)^{m_1}, \ldots, x_r, \left( \frac{2t_r}{1 + (1 - 4t_r)^{1/2}} \right)^{m_r}]
\]

Putting \( \phi_r = m_r \) and \( \beta_r = 1 - \lambda_r - 2n_r \) in equation (3.5), we get (3.6)

\[
\sum_{n_1, \ldots, n_r=0}^{\infty} \frac{(\lambda_1)_{2n_1} \cdots (\lambda_r)_{2n_r} t_1^{n_1} \cdots t_r^{n_r}}{(\lambda_1)_{n_1} \cdots (\lambda_r)_{n_r} n_1! \cdots n_r!} F_D^{(r)}[(\lambda; m_1); (-n_i; m_i); (\gamma; \Psi); x_1, \ldots, x_r]
\]

\[
= \prod_{i=1}^{r} \left( (1 - 4t_i)^{-1/2} \left[ \frac{2}{1 + (1 - 4t_i)^{1/2}} \right]^{\lambda_i-1} \right)
\cdot F_D^{(r)}[(\lambda; m_1); (1 - \lambda_i - 2n_i; m_i); (\gamma; \Psi); x_1, \left( \frac{2t_1}{1 + (1 - 4t_1)^{1/2}} \right)^{m_1}, \ldots, x_r, \left( \frac{2t_r}{1 + (1 - 4t_r)^{1/2}} \right)^{m_r}].
\]
Again, making use of the Mittal [3] operational formula

\[
\sum_{n=0}^{\infty} \frac{1}{n!} T^n_{a-n} \{ f(x) \} = (1 + x)^{a-1} f[x(1 + x)],
\]

and proceeding as above, we get the generating relation

\[
\sum_{n_1, \ldots, n_r=0}^{\infty} (-1)^{n_1+\cdots+n_r} \prod_{i=1}^{r} \frac{(1 - \lambda_i)_{n_i} t_i^{n_i}}{n_i!} \cdot F^{1; \ldots; 2n}_{1; 0; \ldots; 0} \left( \left( \lambda ; m_i \right) : (-n_i ; m_i) \right),
\]

\[
\left( \gamma ; \Psi_i \right) ; x_1, \ldots , x_r \right)
\]

\[
= \prod_{i=1}^{r} (1 + t_i)^{\lambda_i - 1} F^{(r)}_{D} \left( \left( \lambda ; m_i \right) : (\lambda_i - n_i ; 2m_i) \right) ; (\gamma ; \Psi_i) ;
\]

\[
x_1(-t_1(1 + t_1)^{m_1}), \ldots , x_r(-t_r(1 + t_r)^{m_r}).
\]

Next, using the Mittal [3] operational formula

\[
\sum_{n=0}^{\infty} \frac{1}{n!} T^n_{a-2n} \{ f(x) \} = (1 + 4x)^{-1/2} \left[ \frac{2}{1 + (1 + 4x)^{1/2}} \right]^{-a} f \left[ \frac{x + (1 + 4x)^{1/2}}{2} \right],
\]

replacing \( a \) by \( a_1 = \lambda_1, \ldots , a_r = \lambda_r, n \) by \( n_1, \ldots , n_r \) and assuming that \( T_{i,k} \) is the operator \( T_k \) operating on \( t_i \), alone, \( i = 1, 2, \ldots , r \), respectively and putting

\[
f(x) \equiv F^{(r)}_{D} \left( \left( \lambda ; m_i \right) : (\beta_i ; \phi_i) \right) ; (\gamma ; \Psi_i) ; x_1 \left( \frac{27}{4} t_1 \right)^{m_1}, \ldots , \left( \frac{27}{4} t_r \right)^{m_r}
\]

in (3.9) and using (3.9) \( r \)-times we get the following generating relation:

\[
\sum_{n_1, \ldots, n_r=0}^{\infty} \prod_{i=1}^{r} \frac{(1 - \lambda_i)_{2n_i} t_i^{n_i}}{n_i!} \cdot \frac{\lambda_{em,k_i}}{(\gamma)_{em,k_i}} \cdot x_1^{k_1} \cdots x_r^{k_r} \cdot F^{(r)}_{D} \left( \left( \lambda ; m_i \right) : (\beta_i ; \phi_i) \right) ; (\gamma ; \Psi_i) ; x_1 \left( \frac{27}{4} t_1 \right)^{m_1}, \ldots , \left( \frac{27}{4} t_r \right)^{m_r}
\]
which is a generalization of a result due to Shukla [2]. Finally replacing $a$ by $a_i = \lambda_i - 1, \ldots, a_r = \lambda_r - 1, n$ by $n_1, \ldots, n_r$ and assuming that $T_{i,k}$ is the operator $T_k$ operating on $t_i$ alone, $i = 1, 2, \ldots, r$ respectively and putting

$$f(x) \equiv F^{(r)}_D \left[ (\lambda; m_i) : (\beta_i; \phi_i) ; (\gamma; \Psi_i) ; x_1 \left[ \frac{(2 - m)^{2-m}}{(1 - m)^{1-m}} t_i \right] \right]^{m_i}, \ldots,$$

$$x_r \left[ \frac{(2 - m)^{2-m}}{(1 - m)^{1-m}} t_r \right]^{m_r},$$

in (2.1) and using it $r$-times we get the generating relation

$$(3.11) \quad \sum_{n_1, \ldots, n_r = 0} (\lambda_i)_{m_n t_i^{n_r}} \prod_{i=1}^{r} n_i! (\lambda_{i})_{(m-1)n_i} \sum_{k_1, \ldots, k_r = 0} \frac{(\lambda)_{e m, k_i} (\beta_i)_{k_i, \phi_i} \cdots (\beta_r)_{k_r, \phi_r} x_1^{k_1} \cdots x_r^{k_r}}{(\gamma)_{s k, \psi, k} ! \cdots k_r !} \prod_{i=1}^{r} \left[ \frac{(-n_i)_{m, k_i} ((\lambda + m n_i) / (1 - m))_{m, k_i} \cdots}{((\lambda + (m - 1)n_i) / (2 - m))_{m, k_i} \cdots} \frac{((\lambda + mn_i - m) / (1 - m))_{m, k_i} \cdots}{((\lambda + (m - 1)n_i + 1 - m) / (2 - m))_{m, k_i} \cdots} \right] \prod_{i=1}^{r} \frac{1 + v_i} {1 - (m - 1)v_i} \left[ (\lambda; m_i) : (\beta_i; \phi_i) ; (\gamma; \Psi_i) ; x_1 \left[ \frac{(2 - m)^{2-m}}{(1 - m)^{1-m}} t_i (1 + v_i) \right] \right]^{m_i}, \ldots,$$

$$x_r \left[ \frac{(2 - m)^{2-m}}{(1 - m)^{1-m}} t_r (1 + v_r) \right]^{m_r},$$

where $v_i = t_i (1 + v_i)^m, i = 1, 2, \ldots, r,$ which is again a generalization of a result due to Shukla [2].

**References**


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